Undecidability of some modal MTL logics (formerly product logics)

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MTL Kripke-models

$\mathbf{A} = \langle A, \odot, \Rightarrow, \min, 1, 0, \rangle$ a complete MTL algebra (conm. integral bounded prelinear residuated lattices = algebras in the variety generated by all left-continuous t-noms).
Language: $\&, \wedge, \rightarrow, \overline{0}$ plus two unary (modal) symbols $(\Box, \Diamond)$

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Definition

A (crisp) **A** Kripke model $\mathfrak{M}$ is a tripla $\langle W, R, e\rangle$ where:

- $R \subseteq W \times W$ (Rus stands for $\langle u, s\rangle \in R$)
- $e : W \times Var \rightarrow A$ uniquelly extended by:
  - $e(u, \varphi \& \psi) = e(u, \varphi) \odot e(u, \psi)$;
    $e(u, \varphi \rightarrow \psi) = e(u, \varphi) \Rightarrow e(u, \psi)$;
    $e(u, \varphi \wedge \psi) = \min\{e(u, \varphi), e(u, \psi)\}$; $e(e, \overline{0}) = 0$
  - $e(u, \Box\varphi) = inf\{e(s, \varphi) : Rus\}$
  - $e(u, \Diamond\varphi) = sup\{e(s, \varphi) : Rus\}$

Modal MTL logics

C a class of complete MTL-algebras.

- **(Global deduction)**: $\Gamma \Vdash_C \varphi$ iff $[\forall u \in W \; e(u, [\Gamma]) \subseteq \{1\}]$ implies $[\forall u \in W \; e(u, \varphi) = 1]$ for all **A** Kripke models $\mathfrak{M}$ with $\mathbf{A} \in C$.
  $\Gamma \Vdash^f_C \varphi$ for denoting the same relation over finite (i.e., finite W) Kripke models.

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  $\Gamma \Vdash^f_C \varphi$ for denoting the same relation over finite (i.e., finite W) Kripke models.
- **(Local deduction)**: $\Gamma \vdash_{4C} \varphi$ iff $\forall u \in W \; [e(u, [\Gamma]) \subseteq \{1\}$ implies $e(u, \varphi) = 1]$ for all **transitive A** Kripke models $\mathfrak{M}$ with $\mathbf{A} \in C$.
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Undecidability results

For $n < \omega$, a MTL-algebra is n-contractive iff it validates the equation

$$x^n \to x^{n+1} = 1$$

A class of MTL-algebras is non contractive iff, for all n, it contains some non n-contractive algebra.

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Let C be a non contractive class of complete MTL-algebras. For arbitrary $\Gamma \cup \{\varphi\}$ the following are undecidable:

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2. $\Gamma \Vdash^f_C \varphi$ (global deduction)
3. $\Gamma \vdash_{4C} \varphi$
4. $\Gamma \vdash^f_{4C} \varphi$ (local deduction in transitive frames)

Post Correspondence Problem

An instance of the PCP is a list of pairs $\langle \mathbf{v_1}, \mathbf{w_1} \rangle \dots \langle \mathbf{v_n}, \mathbf{w_n} \rangle$ where $\mathbf{v_i}, \mathbf{w_i}$ are numbers in base $s \geq 2$.

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- $\mathbf{a}, \mathbf{b}$ numbers in base $s \implies \mathbf{ab} = \mathbf{a} \cdot s^{\|\mathbf{b}\|} + \mathbf{b}$, where $\| \mathbf{b} \|$ is the length of $\mathbf{b}$ (in base s).

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- we can exploit the conjunction operation to express concatenation (using powers over some y "non-contractive")

The global modal logic case

Given a PCP instance P there is a finite set of formulas $\Gamma_g(P) \cup \{\varphi_g\}$ such that

$$P \text{ is SAT} \iff \Gamma_g(P) \not\Vdash_C \varphi_g$$

Moreover $\Gamma_g(P) \Vdash_C \varphi_g \iff \Gamma_g(P) \Vdash_C^f \varphi_g$.

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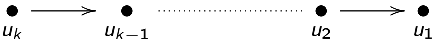
- Proving $\Longrightarrow$ will not be hard (constructing a model using the solution of P).
- Idea for $\Longleftarrow$: if $\Gamma_g(P) \not\Vdash \varphi_g$ then it happens in u_k of a particular structure shaped like



The global case: formulas

Variables used: $\mathcal{V} = \{x, y, z, v, w\}$. y, z, are control variables; x stores information on the index of the added word; v, w store information on the concatenation.

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Formulas of $\Gamma_g(P)$:

- $(\neg\Box\bar{0}) \rightarrow (\Box p \leftrightarrow \Diamond p)$ for each $p \in \mathcal{V}$:

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Lemma

If $\Gamma_g(P) \not\Vdash_C \psi$ (for arbitrary ψ in $\mathcal{V}$) then there is a C Kripke model $\mathfrak{M}$ with $W = \{u_i : i \in \omega\}$ or $W = \{u_i : i \leq k\}$ and $R = \{\langle u_i, u_{i+1}\rangle\}$ such that

- $\mathfrak{M}$ is a model for $\Gamma_g(P)$ and
- $e(u_1, \psi) < 1$

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- $(x \leftrightarrow z^i) \to (v \leftrightarrow (\Box v)^{s^{\|\mathbf{v}_i\|}} \& y^{\mathbf{v}_i})$ for each $1 \leq i \leq n$:

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- $(x \leftrightarrow z^i) \rightarrow (v \leftrightarrow (\Box v)^{s^{\|\mathbf{v}_i\|}} \& y^{\mathbf{v}_i})$ for each $1 \leq i \leq n$: (information on the concatenation of $\mathbf{v}s$)

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Let $\varphi_g = (v \leftrightarrow w) \rightarrow ((v \rightarrow v\&y) \vee (w \rightarrow w\&y) \vee (z^{n-1} \rightarrow z^n))$.

The global case: main result

Lemma

Let $\mathfrak{M}$ with $W = \{u_i : 1 \leq i \leq \kappa\}$ and $R = \{\langle u_{i+1}, u_i \rangle : 1 \leq i < \kappa\}$ be a model of $\Gamma_g(P)$ such that $e(u_\kappa, \varphi_g) < 1$. Then

1. $\kappa < \omega$ (i.e, the model is finite)

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2. $\alpha_z^n < \ldots < \alpha_z$ (determining indexes from 1 to n)

   follows from $e(u_\kappa, z^n) < e(u_\kappa, z^{n-1})$

The global case: main result

3. for all $1 \leq j \leq \kappa$, $e(u_j, v) = \alpha_y^{v_{i_1} \cdots v_{i_j}}$ and $e(u_j, w) = \alpha_y^{w_{i_1} \cdots w_{i_j}}$ for $e(u_j, x) = \alpha_z^{i_j}$ for $1 \leq j \leq k$.

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4. let $a = \max\{v_{i_1} \cdots v_{i_\kappa}, w_{i_1} \cdots w_{i_\kappa}\}$. For any $1 \leq b < c \leq a$ it holds $\alpha_y^c < \alpha_y^b$.

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   otherwise, $e(u_\kappa, v \leftrightarrow w) \leq \alpha_y$ and we know $e(u_\kappa, v \rightarrow v \& y) \geq \alpha_y$ (contradicting $e(u_\kappa, \varphi_g) < 1$).

From P to a model and back

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  - $e(u_i, z) = \alpha_y^m$ with m depending on $v_{i_1} \cdots v_{i_k}$ and $w_{i_1} \cdots w_{i_k}$, $(\alpha_y^m = \min_{1 \leq j \leq k} \alpha_y^{v_{i_1} \cdots v_{i_k}} \leftrightarrow \alpha_y^{v_{i_1} \cdots v_{i_{k-1}} \cdot v_j})$

From P to a model and back

- If $\Gamma_g(P) \not\Vdash_C^{(f)} \varphi_g$ in u_k of a model $\mathfrak{M}$ as the one from before we can naturally get a solution for P.
- If $i_1, \ldots, i_k$ is a solution for P, then $\Gamma_g(P) \not\Vdash_C^{(f)} \varphi_g$ in u_k of the model $\mathfrak{M} = \langle \{u_1, \ldots, u_k\}, \{\langle u_k, u_{k-1}\rangle, ..., \langle u_2, u_1\rangle\}, e\rangle$ with
  - $e(u, y) = \alpha_y \in A(\in C)$ such that α_y (and so, **A**) is non r-contractive for r depending on n and $v_{i_1} \cdots v_{i_k}$,
  - $e(u_j, v) = \alpha_y^{\mathbf{v_{i_1} \cdots v_{i_j}}}$ (analogously for w),
  - $e(u_i, z) = \alpha_y^m$ with m depending on $v_{i_1} \cdots v_{i_k}$ and $w_{i_1} \cdots w_{i_k}$, $(\alpha_y^m = \min_{1 \leq j \leq k} \alpha_y^{v_{i_1} \cdots v_{i_k}} \leftrightarrow \alpha_y^{v_{i_1} \cdots v_{i_{k-1}} \cdot v_j})$
  - $e(u_j, x) = \alpha_z^{i_j}$ (observe $e(u_j, x \leftrightarrow z^r)$ for $1 \leq r \leq n$ is either 1 (if $r = i_j$) or is $\leq \alpha_z$).

The local modal logic case

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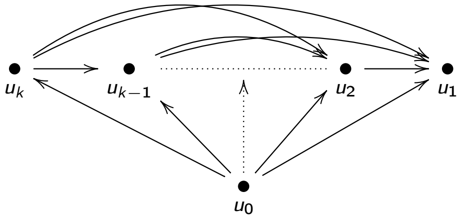
We now work towards structures with the form



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The construction of a model $\mathfrak{M}$ from a solution of P and viceversa are similar to the ones from the global case.

Thank you!