

Axiomatizing modal fixpoint logics

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(largely joint work with Enqvist, Seifan, Santocanale, Schröder, . . .)

Overview

- ▶ Introduction
- ▶ Obstacles
- ▶ A general result
- ▶ A general framework
- ▶ Frame conditions
- ▶ Conclusions

Example

- ▶ Add master modality $\langle * \rangle$ to the language ML of modal logic
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- ▶ Variant (PDL): $\langle \alpha^* \rangle \varphi := \mu x. \varphi \vee \langle \alpha \rangle x$

More examples

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- ▶ $C\varphi := \varphi \wedge \bigwedge_i K_i\varphi \wedge \bigwedge_i K_i(C\varphi) \wedge \dots$
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 $C\varphi \equiv \varphi \wedge \bigwedge_i K_i C\varphi$
 $C\varphi := \nu x. \varphi \wedge \bigwedge_i K_i x$

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- ▶ Motivation 2: generally **nice computational properties**
- ▶ Combined: many **applications** in process theory, epistemic logic, \dots
- ▶ Interesting mathematical theory:
 - ▶ interesting mix of algebraic|coalgebraic features
 - ▶ connections with theory of **automata** on infinite objects
 - ▶ **game-theoretical** semantics
 - ▶ interesting meta-logic

General Program

Understand modal fixpoint logics by studying the interaction between

- combinatorial
- algebraic and
- coalgebraic

aspects

Here: consider axiomatization problem

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- ▶ a **least (pre-)fixpoint axiom**:

$$\varphi(\mu p.\varphi) \vdash \mu p.\varphi$$

- ▶ Park's **induction rule**

$$\frac{\varphi(\psi) \vdash \varphi}{\mu p.\varphi \vdash \psi}$$

(Here $\alpha \vdash_K \beta$ abbreviates $\vdash_K \alpha \rightarrow \beta$)

Axiomatization results for modal fixpoint logics

- ▶ LTL: Gabbay et alii (1980)
- ▶ PDL: Kozen & Parikh (1981)
- ▶ μ ML (aconjunctive fragment): Kozen (1983)
- ▶ CTL: Emerson & Halpern (1985)
- ▶ μ ML: Walukiewicz (1993/2000)
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So what is the **problem**?

Axiomatization problem

Questions (2015)

- ▶ How to generalise these results to restricted frame classes?
- ▶ How to generalise results to similar logics, eg, the monotone μ -calculus?
- ▶ Does completeness transfer to fragments of μ ML? (Ex: game logic)
- ▶ What about proof theory?

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Compared to basic modal logic

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Compared to basic modal logic

- ▶ there are **no sweeping general results** such as Sahlqvist's theorem
- ▶ there is **no comprehensive completeness theory** (duality, canonicity, filtration, ...)

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 - ▶ $(m, n)R(m', n')$ iff $m' = m + 1$ and $n' = n$
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- ▶ Logic $\mathbf{KG} := \mathbf{K} +$
 - ▶ functionality: $\diamond_R p \leftrightarrow \square_R p$ and $\diamond_U p \leftrightarrow \square_U p$
 - ▶ confluence: $\diamond_R \square_U p \rightarrow \square_U \diamond_R p$

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 - ▶ **Proof:** Use recurrent tiling problem to show that
 - ▶ the $\diamond_R, \diamond_U, \langle * \rangle$ -logic of $Fr(\mathbf{KG})$ is not recursively enumerable

Obstacle 2: compactness failure

- ▶ Example: $\langle * \rangle p := \bigvee_{n \in \omega} \diamond^n p$
 - ▶ $\{\langle * \rangle p\} \cup \{\square^n \neg p \mid n \in \omega\}$ is **finitely satisfiable but not satisfiable**

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- ▶ Fixpoint logics have **no nice Stone-based duality**

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- ▶ obstacle 3b: **fixpoint alternations** cause intricate combinatorics

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- ▶ restrict language to **fixpoints of simple formulas** (avoid alternation)
- ▶ allow alternation, but develop **suitable combinatorial framework**

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- ▶ Obtain language ML_γ :

$$\varphi ::= p \mid \neg p \mid \perp \mid \top \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \wedge \varphi_2 \mid \diamond_i \varphi \mid \square_i \varphi \mid \#_\gamma(\vec{\varphi})$$

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- ▶ Examples: CTL, LTL, (PDL), ...

Flat Modal Fixpoint Logics: Kripke Semantics

- ▶ Kripke frame $S = \langle S, R \rangle$ with $R \subseteq S \times S$.
- ▶ Complex algebra: $S^+ := \langle \wp(S), \emptyset, S, \sim_S, \cup, \cap, \langle R \rangle \rangle$,
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- ▶ Every modal formula $\varphi(p_1, \dots, p_n)$ corresponds to a **term function**

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- ▶ **Kripke \sharp -algebra** $S^\sharp := \langle \wp(S), \emptyset, S, \sim_S, \cup, \cap, \langle R \rangle, \sharp^S \rangle$.

Candidate Axiomatization

\mathbf{K}_γ := \mathbf{K} extended with

- ▶ prefixpoint axiom:

$$\gamma(\#(\vec{\varphi}), \vec{\varphi}) \vdash \#(\vec{\varphi})$$

- ▶ Park's induction rule:

from $\gamma(\psi, \vec{\varphi}) \vdash \psi$ infer $\#_\gamma(\vec{\varphi}) \vdash \psi$.

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$$\sharp(\vec{b}) = LFP.\gamma_{\vec{b}}^A,$$

where $\gamma_{\vec{b}}^A : A \rightarrow A$ is given by $\gamma_{\vec{b}}^A(a) := \gamma^A(a, \vec{b})$.

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- ▶ **Axiomatically**: modal \sharp -algebras satisfy
 - ▶ $\gamma(\sharp(\vec{y}), \vec{y}) \leq \sharp(\vec{y})$
 - ▶ if $\gamma(x, \vec{y}) \leq x$ then $\sharp(\vec{y}) \leq x$.
- ▶ Completeness for flat fixpoint logics: $\text{Equ}(\text{MA}_{\sharp}) \stackrel{?}{=} \text{Equ}(\text{KA}_{\sharp})$

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- ▶ Completeness for flat fixpoint logics: $\text{Equ}(\text{MA}_{\sharp}) \stackrel{?}{=} \text{Equ}(\text{KA}_{\sharp})$
- ▶ Two key concepts:

Flat Modal Fixpoint Logics: Algebraic completeness proof

- ▶ **Modal \sharp -algebra**: $A = \langle A, \perp, \top, \neg, \wedge, \vee, \diamond, \sharp \rangle$ with $\sharp : A^n \rightarrow A$ satisfying

$$\sharp(\vec{b}) = LFP.\gamma_{\vec{b}}^A,$$

where $\gamma_{\vec{b}}^A : A \rightarrow A$ is given by $\gamma_{\vec{b}}^A(a) := \gamma^A(a, \vec{b})$.

- ▶ **Axiomatically**: modal \sharp -algebras satisfy
 - ▶ $\gamma(\sharp(\vec{y}), \vec{y}) \leq \sharp(\vec{y})$
 - ▶ if $\gamma(x, \vec{y}) \leq x$ then $\sharp(\vec{y}) \leq x$.
- ▶ Completeness for flat fixpoint logics: $\text{Equ}(\text{MA}_{\sharp}) \stackrel{?}{=} \text{Equ}(\text{KA}_{\sharp})$
- ▶ Two key concepts:
 - ▶ **constructiveness**
 - ▶ **\mathcal{O} -adjointness**

Constructiveness

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Let A be a countable, residuated, modal \sharp -algebra.

If A is constructive, then A can be embedded in a Kripke \sharp -algebra.

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Proof

Via a step-by-step construction/generalized Lindenbaum Lemma.

Alternatively, use Rasiowa-Sikorski Lemma.

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Theorem (Santocanale 2005)

If $f : A \rightarrow A$ is a finitary \mathcal{O} -adjoint, then $LFP.f$, if existing, is constructive.

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- ▶ Counterexamples: $\diamond(x \wedge \diamond x)$, $\diamond x \wedge \square \diamond x$

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Theorem (Santocanale & YV 2010)

Untied formulas are finitary \mathcal{O} -adjoints.

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- ▶ Schröder & YV have similar results for wider coalgebraic setting.

Overview

- ▶ Introduction
- ▶ Obstacles
- ▶ A general result
- ▶ A general framework
- ▶ Frame conditions
- ▶ Conclusions

The modal μ -calculus

- ▶ [+] natural extension of basic modal logic with fixpoint operators
- ▶ [+] expressive: LTL, CTL, PDL, CTL*, ... $\subseteq \mu\text{ML}$
- ▶ [+] good computational properties
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Most results on μML use **automata** ...

Logic & Automata

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Automata in Logic

- ▶ long & rich history (Büchi, Rabin, ...)
- ▶ mathematically interesting theory
- ▶ many practical applications
- ▶ automata for μ ML:
 - ▶ Janin & Walukiewicz (1995): μ -automata (nondeterministic)
 - ▶ Wilke (2002): [modal automata](#) (alternating)

Modal automata

Fix a set X of proposition letters; PX is a set of colours

- ▶ A modal automaton is a triple $\mathbb{A} = (A, \Theta, Acc)$, where
 - ▶ A is a finite set of states
 - ▶ $\Theta : A \times PX \rightarrow 1ML(A)$ is the transition map
 - ▶ $Acc \subseteq A^\omega$ is the acceptance condition

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 - ▶ Given $\rho \in A^\omega$, $Inf(\rho) := \{a \in A \mid a \text{ occurs infinitely often in } \rho\}$
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- ▶ Our approach: **automata are formulas**

One-step logic 1ML

- ▶ Let A be a set of variables with $A \cap X = \emptyset$
- ▶ One-step formulas: $\diamond(a \wedge b)$, $\Box a \wedge \diamond b$, \top , $\diamond \perp, \dots$
- ▶ A one-step model is a pair (U, m) with $m : U \rightarrow PA$ a marking
 - ▶ write $U, m, u \Vdash^0 a$ if $a \in m(u)$

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$$\begin{aligned}\alpha &::= \diamond\pi \mid \Box\pi \mid \perp \mid \top \mid \alpha \vee \alpha \mid \alpha \wedge \alpha \\ \pi &::= a \in A \mid \perp \mid \top \mid \pi \vee \pi \mid \pi \wedge \pi\end{aligned}$$

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- ▶ One-step semantics interprets $1ML(A)$ over one-step models, e.g.
 - ▶ $(U, m) \Vdash^1 \Box a$ iff $\forall u \in U. u \Vdash^0 a$
 - ▶ $(U, m) \Vdash^1 \diamond(a \wedge b)$ iff $\exists u \in U. u \Vdash^0 a \wedge b$

Acceptance game

- Represent Kripke model as pair $\mathbb{S} = (S, \sigma)$ with $\sigma : S \rightarrow PX \times PS$

Acceptance game $\mathcal{A}(\mathbb{A}, \mathbb{S})$ of $\mathbb{A} = \langle A, \Theta, Acc \rangle$ on $\mathbb{S} = \langle S, \sigma \rangle$:

Position	Player	Admissible moves
$(a, s) \in A \times S$	\exists	$\{m : \sigma_R(s) \rightarrow PA \mid \sigma(s), m \Vdash^1 \Theta(a)\}$
$m : S \rightarrow PA$	\forall	$\{(b, t) \mid b \in m(t)\}$

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Winning conditions:

- ▶ finite matches are lost by the player who gets stuck,
- ▶ infinite matches are won as specified by the **acceptance condition**:
 - ▶ match $\pi = (a_0, s_0)m_0(a_1, s_1)m_1 \dots$ induces list $\pi_A := a_0 a_1 a_2 \dots$
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Definition (\mathbb{A}, a) **accepts** (\mathbb{S}, s) if $(a, s) \in Win_{\exists}(\mathcal{A}(\mathbb{A}, \mathbb{S}))$.

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 - by the interaction of combinatorics and dynamics

Automata & ...

Theorem

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As a corollary, we may apply proof-theoretic concepts to automata

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 - ▶ **monotonicity** rule for \diamond : $a \leq b / \diamond a \leq \diamond b$
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$$\vdash_{\mathbf{H}} \alpha \leq \alpha' \text{ iff } \models^1 \alpha \leq \alpha'.$$

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 - ▶ **normality** ($\diamond \perp \leq \perp$) and **additivity** ($\diamond(a \vee b) \leq \diamond a \vee \diamond b$) axioms
- ▶ A derivation system \mathbf{H} is **one-step sound and complete** if

$$\vdash_{\mathbf{H}} \alpha \leq \alpha' \text{ iff } \models^1 \alpha \leq \alpha'.$$

- ▶ For more on this, check the literature on coalgebra (Cîrstea, Pattinson, Schröder, ...)

General result

Theorem Assume that

- ▶ \mathcal{L} is a one-step language with an adequate disjunctive base
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'De- and re-constructing' Walukiewicz' proof – automata in leading role

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- ▶ linear time μ -calculus, k -successor μ -calculus, standard modal μ -calculus, graded μ -calculus, monotone modal μ -calculus, game μ -calculus, ...

Overview

- ▶ Introduction
- ▶ Obstacles
- ▶ A general result
- ▶ A general framework
- ▶ **Frame conditions**
- ▶ Conclusions

Frame conditions

Conjecture Let \mathbf{L} be an extension of \mathbf{K}_Γ or \mathbf{K}_μ with an axiom set Φ such that each $\varphi \in \Phi$

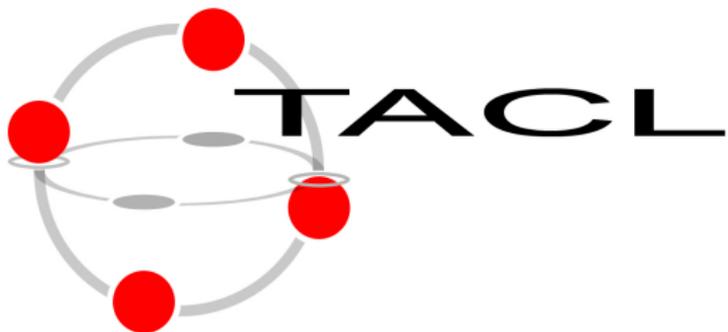
- ▶ is canonical
- ▶ corresponds to a **universal** first-order frame condition.

Then \mathbf{L} is sound and complete for the class of frames satisfying Φ .

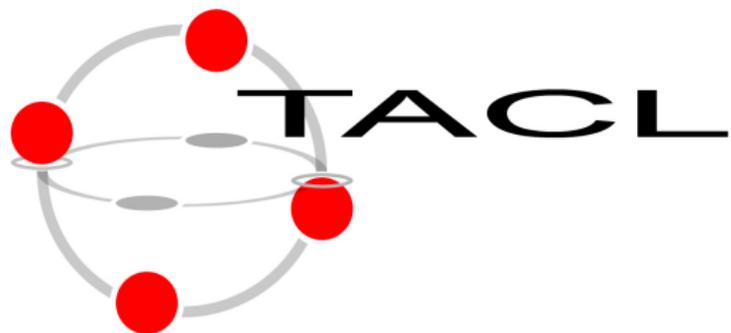
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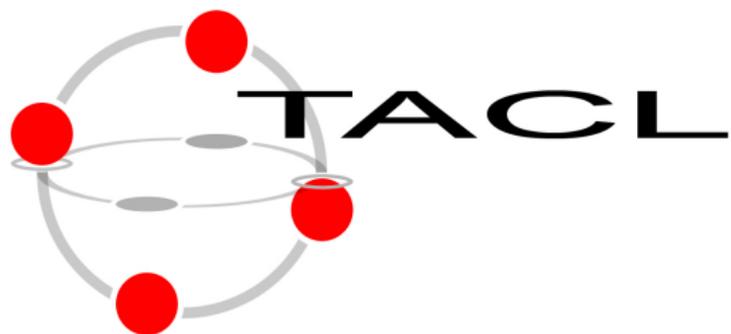


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TOPOLOGY, ALGEBRA AND CATEGORIES IN LOGIC 2017

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2017 June 20–24 : TACL School

2017 June 26–30 : TACL Conference

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- ▶ perspective for bringing automata into proof theory

Future work

- ▶ prove conjecture!
- ▶ completeness for fragments of μ ML (game logic!)
 - ▶ many μ ML-fragments have interesting automata-theoretic counterparts!
- ▶ interpolation for fixpoint logics (PDL!)
- ▶ fixpoint logics on non-boolean basis
 - ▶ non-boolean automata?
- ▶ proof theory for modal automata
- ▶ further explore notion of \mathcal{O} -adjointness
- ▶ ...

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- ▶ L. Schröder & YV. [Completeness for flat coalgebraic fixpoint logic](#) submitted (short version appeared in CONCUR 2010)
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<http://staff.science.uva.nl/~yde>

THANK YOU!