A uniform way to build strongly perfect MTL-algebras via Boolean algebras and prelinear semihoops

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(with Stefano Aguzzoli and Tommaso Flaminio)

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Introduction

A MTL-algebra is a structure $\mathbf{A} = (A, \odot, \to, \wedge, \vee, \bot, \top)$ where:

- $(A, \wedge, \vee, \bot, \top)$ is a bounded distributive lattice,
- $(A, \odot, \top)$ is a commutative monoid,
- $x \odot y \leq z \Leftrightarrow z \leq x \to y$ holds for every $x, y, z \in A$,
- $(x \to y) \vee (y \to x) = \top$ holds for every $x, y \in A$.

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In every MTL-algebra we can define further operations and abbreviations:

$$\neg x = x \to \bot, \quad x \oplus y = \neg(\neg x \odot \neg y), \quad x^2 = x \odot x, \quad 2x = x \oplus x.$$

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BL-algebras are MTL algebras that satisfy divisibility:

$$x * (x \rightarrow y) = y * (y \rightarrow x).$$

Introduction

A *strongly perfect MTL-algebra* (SBP_0-algebra) is any MTL-algebra satisfying:

(DL) $(2x)^2 = 2(x^2)$.

(N) $\neg(x)^2 \to (\neg\neg x \to x) = 1$,

The class of SBP_0 forms a variety denoted by $\mathbb{SBP}_0$.

We will denote $\mathbb{BP}_0 = \mathbb{MTL} + (DL)$ (i.e. the variety generated by *perfect* MTL-algebras).

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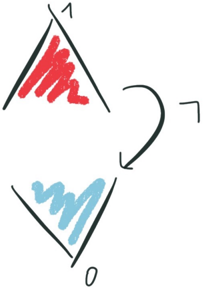
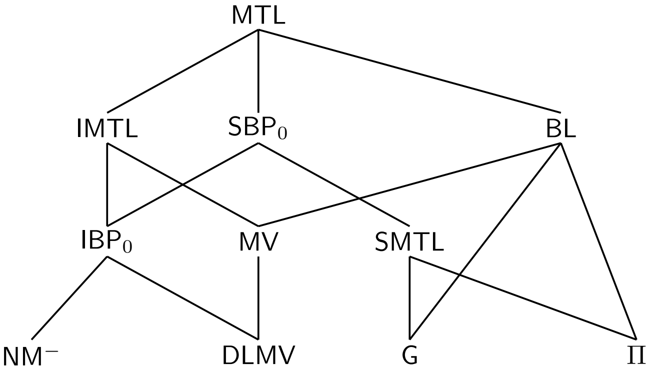


Introduction

Notable subvarieties of SBP_0 algebras are:

- Pseudocomplemented MTL-algebras SMTL: MTL $+ \quad x \wedge \neg x = 0$
  - Product algebras Π : BL $+ \quad \neg x \vee ((x \to x \cdot y) \to y) = 1$
  - Gödel algebras G : BL $+ \quad x \cdot x = x$
- Involutive BP_0-algebras IBP_0 : $\mathsf{SBP}_0 \quad + \quad \neg\neg x = x$
  - The variety generated by perfect MV-algebras
    DLMV : $\mathsf{IBP}_0 \quad + \quad x * (x \to y) = y * (y \to x)$
  - The variety generated by the nilpotent minimum algebra $[0, 1]\backslash\{1/2\}$
    NM^- : IDL $+ \quad \neg(x^2) \vee (x \to x^2) = 1.$

Introduction



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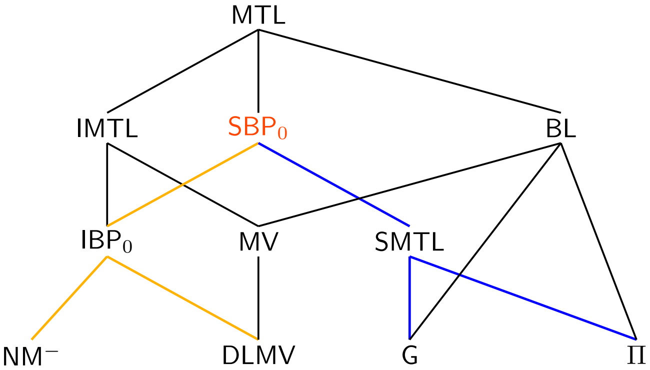

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A prelinear semihoop is an algebra $\mathbf{H} = (H, *, \to, \wedge, \vee, 1)$ such that:

- $(H, *, 1)$ is a commutative monoid,
- $(H, \wedge, \vee, 1)$ is a lattice with top element 1,
- $(*, \to)$ forms a residuated pair,
- $\mathbf{H}$ is prelinear: $(x \to y) \vee (y \to x) = 1$, for all $x, y \in H$

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Notable subvarieties:

Basic hoops :	PSH	+	$x * (x \to y) = y * (y \to x)$
Gödel hoops :	BH	+	$x = x^2$
Wajsberg hoops :	BH	+	$(x \to y) \to y = (y \to x) \to x$
Cancellative hoops :	BH	+	$x \to (x * y) = y$

CH-Triples and product algebras

[Montagna - U., 2015]: The category $\mathbb{P}$ of product algebras is equivalent to a category whose objects are triples $(\mathbf{B}, \mathbf{C}, \vee_e)$, where $\mathbf{B}$ is a Boolean algebra, $\mathbf{C}$ is a cancellative hoop and $\vee_e : B \times C \to C$ satisfies suitable properties.

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Key idea Directly indecomposable product algebras are of the kind $\mathbf{2} \oplus \mathbf{C}$ [Cignoli, Torrens].

Directly indecomposable SMTL and IBP_0 algebras

Any directly indecomposable SMTL algebra $\mathbf{A}$ is a lifting of a prelinear semihoop $\mathbf{H}$, $\mathbf{A} = \mathbf{2} \oplus \mathbf{H}$:

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Any directly indecomposable IBP_0 algebra $\mathbf{A}$ is a disconnected rotation of a prelinear semihoop $\mathbf{H}$, $A = \{0, 1\} \times H$:

Directly indecomposable SBP_0 algebras

More in general, every directly indecomposable SBP_0 algebra can be obtained starting from a prelinear semihoop $\mathbf{H}$, using a a weakening of Cignoli-Torrens dl-admissible operator δ.

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We shall call a map $\delta : H \to H$ w-admissible iff for all $a, b \in H$:

$$
\begin{aligned}
a \to \delta(a) &= 1, & \delta(\delta(a)) &= a, \\
\delta(a \to b) &\leq a \to \delta(b), & \delta(a * b) &= \delta(\delta(a) * \delta(b)), \\
\delta(a \wedge b) &= \delta(a) \wedge \delta(b), & \delta(a \vee b) &= \delta(a) \vee \delta(b).
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Observation: the weakened condition allows to get rid of Glivenko equation $\neg\neg(\neg\neg x \to x) = 1$.

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Examples

- $\delta_D(a) = a$ for all $a \in H$.
- $\delta_L(a) = 1$ for all $a \in H$.

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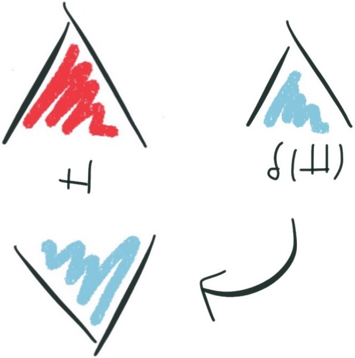

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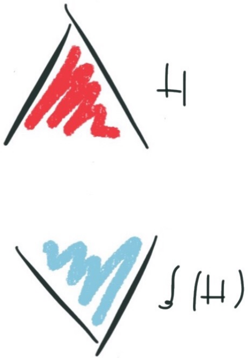
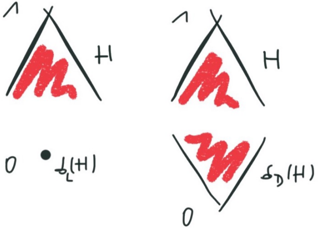


Directly indecomposable SBP_0 algebras

In particular, with δ_L and δ_D we obtain respectively directly indecomposable SMTL and IBP_0 algebras.

Indeed, $\delta_L(H) = \{1\}$ and $\delta_D(H) = H$.



Directly indecomposable SMTL and IBP_0 algebras

Every $a \in \mathbf{A}$ d.i. SMTL-algebra is such that $a = b \wedge c$, where $b \in \{0, 1\}$, $c \in H$.

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Every $a \in \mathbf{A}'$ d.i. IBP_0-algebra is such that $a = (b \wedge c) \vee (\neg b \wedge \neg c)$, where $b \in \{0, 1\}$, $c \in H$.

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Since this equation hold in any directly indecomposable SBP_0 algebra, it holds for any algebra of the variety.

SBP_0 algebras decomposition

Let $\mathbf{A}$ be a SBP_0 algebra, then to each $a \in A$ we can associate a pair (b, c) where b is *boolean* and c is an element of the greatest prelinear sub-semihoop of $\mathbf{A}$.

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- $B_A = \{x \in A \,|\, x \vee \neg x \;= 1\}$ is the dominium of the greatest Boolean subalgebra, or the Boolean skeleton, of $\mathbf{A}$.
- $H_A = \{x \in A \,|\, x > \neg x\}$ is the dominium of the greatest prelinear semihoop contained in $\mathbf{A}$, that is exactly $Rad(\mathbf{A})$.

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But a pair $(\mathbf{B}, \mathbf{H})$ does not uniquely determine $\mathbf{A}$.

PSH-triples

A *prelinear semihoop triple*, a PSH-triple, is a triple $(\mathbf{B}, \mathbf{H}, \vee_e)$ where $\mathbf{B}$ is a Boolean algebra, $\mathbf{H}$ is a prelinear semihoop such that $B \cap H = \{1\}$, and $\vee_e$ is a map from $\mathbf{B} \times \mathbf{H}$ into $\mathbf{H}$ such that:

(V1) For fixed $b \in B$ and $c \in H$:
the map $h_b(x) = b \vee_e x$ is an endomorphism of $\mathbf{H}$,
the map $k_c(x) = x \vee_e c$ is a lattice homomorphism from $\mathbf{B}$ into $\mathbf{H}$.

(V2) h_0 is the identity on $\mathbf{H}$,
h_1 is constantly equal to 1.

(V3) For all $b, b' \in B$ and for all $c, c' \in H$,
$h_b(c) \vee h_{b'}(c') = h_{b \vee b'}(c \vee c') = h_b(h_{b'}(c \vee c'))$.

The category of PSH-triples

A good morphism pair from a PSH-triple $(\mathbf{B}, \mathbf{H}, \vee_e)$ to another PSH-triple $(\mathbf{B}', \mathbf{H}', \vee_e')$ is a pair (h, k) where:

- h is a homomorphism from $\mathbf{B}$ to $\mathbf{B}'$,
- k is a homomorphism from $\mathbf{H}$ to $\mathbf{H}'$,
- for all $x \in B$ and $y \in H$, $k(x \vee_e y) = h(x) \vee_e' k(y)$.

The category $\mathcal{T}_{\mathbb{PSH}}$ of PSH-triples has PSH-triples as objects and good morphism pairs as morphisms, with composition defined componentwise: $(h, k) \circ (h', k') = (h \circ h', k \circ k')$.

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We can define a functor Φ from the category of SBP$_0$-algebras $\mathbb{SBP}_0$ to $\mathcal{T}_{\mathbb{PSH}}$ as follows:

- $\Phi(\mathbf{A}) = (\mathbf{B_A}, \mathbf{H_A}, \vee)$
- $\Phi(f) = (f_{|_{B_A}}, f_{|_{H_A}})$

Inverting Φ: building a SBP_0-algebra

Let $(\mathbf{B}, \mathbf{H}, \vee_e)$ be a PSH-triple, and let δ be a w-admissible operator on $\mathbf{H}$. We define $(b, c) \sim_e (b', c')$ iff

$$b = b', \ \neg b \vee_e c = \neg b \vee_e c' \quad \text{and} \quad b \vee_e \delta(c) = b \vee_e \delta(c')$$

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In a SMTL algebra, a pair (b, c) intuitively represent the element of the SBP_0 algebra $b \wedge c$.

Hence, for instance, all pairs of the kind $(0, c)$ for any $c \in H$ represent the same element.

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where, for all $(b, c), (b', c') \in B \times H$:

$(b, c) \odot (b', c') =$
$(b \wedge b', h_{b \vee b'}(1) \wedge h_{b \vee \neg b'}(c' \to c) \wedge h_{\neg b \vee b'}(c \to c') \wedge h_{\neg b \vee \neg b'}(c * c'))$;

$(b, c) \Rightarrow (b', c') =$
$(b \to b', h_{b \vee b'}\delta(c' \to c) \wedge h_{b \vee \neg b'}(1) \wedge h_{\neg b \vee b'}\delta(c * c') \wedge h_{\neg b \vee \neg b'}(c \to c'))$;

$(b, c) \sqcap (b', c') =$
$(b \wedge b', h_{b \vee b'}(c \vee c') \wedge h_{b \vee \neg b'}(c) \wedge h_{\neg b \vee b'}(c') \wedge h_{\neg b \vee \neg b'}(c \wedge c'))$;

$(b, c) \sqcup (b', c') =$
$(b \vee b', h_{b \vee b'}(c \wedge c') \wedge h_{b \vee \neg b'}(c') \wedge h_{\neg b \vee b'}(c) \wedge h_{\neg b \vee \neg b'}(c \vee c'))$.

Where $h_b : H \to H$, $h_b(c) = b \vee_e c$ for all $b \in B$ and $c \in H$.

Inverting Φ: functor Ξ^{δ}

Theorem

$\mathbf{B} \otimes_{e}^{\delta} \mathbf{H}$ *is a SBP*$_0$*-algebra, for every* δ *w-admissible operator on* $\mathbf{H}$.

Inverting Φ: functor Ξ^δ

Theorem

$\mathbf{B} \otimes_e^\delta \mathbf{H}$ *is a SBP$_0$-algebra, for every δ w-admissible operator on $\mathbf{H}$.*

Moreover, as expected:

- $\mathbf{B} \otimes_e^{\delta_L} \mathbf{H}$ is a SMTL-algebra,
- $\mathbf{B} \otimes_e^{\delta_D} \mathbf{H}$ is a IBP$_0$-algebra.

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We define functor Ξ^{δ_L} (or Ξ^{δ_D}) from $\mathcal{T}_{\mathbb{PSH}}$ into $\mathbb{SMTL}$ (or $\mathbb{IBP}_0$, respectively) as follows:

- $\Xi^{\delta_{L,D}}(\mathbf{B}, \mathbf{H}, \vee_e) = \mathbf{B} \otimes_e^{\delta_{L,D}} \mathbf{C}$
- $\Xi^{\delta_{L,D}}(h, k)([b, c]) = [h(b), k(c)]$.

Categorical equivalences

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Let $\mathbb{SMTL}_{\mathbb{H}}$ and $\mathbb{IBP}_{0\mathbb{H}}$ be the full subcategory respectively of $\mathbb{SMTL}$ and $\mathbb{IBP}_0$ consisting of algebras $\mathbf{A}$ such that the maximum sub-semihoop $\mathbf{H}_{\mathbf{A}} \in \mathbb{H}$.

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Suitably restricting the functors, we can prove the following.

Theorem

Given any $\mathbb{H}$ subvariety of $\mathbb{PSH}$, it holds:

- (Φ, Ξ^{δ_L}) *provide a categorical equivalence between $\mathbb{SMTL}_{\mathbb{H}}$ and $\mathcal{T}_{\mathbb{H}}$.*
- (Φ, Ξ^{δ_D}) *provide a categorical equivalence between $\mathbb{IBP}_{0\mathbb{H}}$ and $\mathcal{T}_{\mathbb{H}}$.*

Corollary

For every $\mathbb{H}$ subvariety of $\mathbb{PSH}$, $\mathbb{SMTL}_{\mathbb{H}}$ and $\mathbb{IBP}_{0\mathbb{H}}$ are categorically equivalent.

Special cases

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CH-Triples Let $\mathbb{H}$ be the variety of cancellative hoops.

Let $\mathbf{B}$ be a Boolean algebra and $\mathbf{C} \in \mathbb{H}$.

Then $\mathbf{B} \otimes_e^{\delta_L} \mathbf{C}$ is a product algebra, $\mathbf{B} \otimes_e^{\delta_D} \mathbf{C}$ is a DLMV-algebra, hence the category of product algebras is equivalent to the category of DLMV algebras $\mathbb{DLMV}$.

Special cases

PSH-Triples Let $\mathbb{H}$ be the variety of prelinear semihoops.
We have that $\mathbb{SMTL}$ is categorically equivalent to $\mathbb{IBP}_0$.

CH-Triples Let $\mathbb{H}$ be the variety of cancellative hoops.

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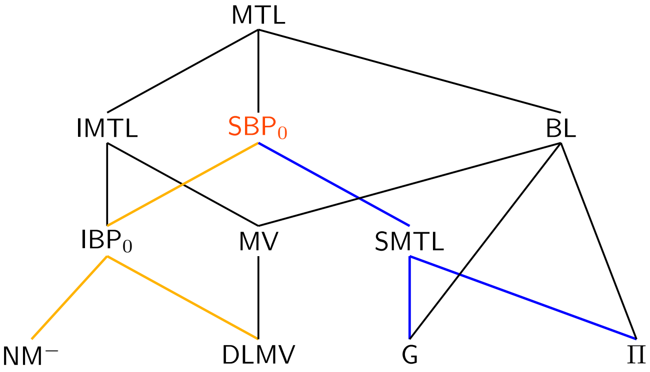
GH-Triples Let $\mathbb{H}$ be the variety of Gödel hoops.

Let $\mathbf{B}$ be a Boolean algebra and $\mathbf{H} \in \mathbb{H}$.

$\mathbf{B} \otimes_e^{\delta_L} \mathbf{H}$ is a Gödel algebra, $\mathbf{B} \otimes_e^{\delta_D} \mathbf{H}$ is a NM$^-$-algebra, and the category of Gödel algebras $\mathbb{G}$ is equivalent to $\mathbb{NM}^-$.

Subvarieties of MTL-algebras



Reaching all strongly perfect MTL-algebras

Fix any $\mathbb{H}$ subvariety of $\mathbb{PSH}$, and let $\mathcal{Q}_{\mathbb{H}}$ be the following category:

- The objects are quadruples $(\mathbf{B}, \mathbf{H}, \vee_e, \delta)$ where $\mathbf{H} \in \mathbb{H}$, $(\mathbf{B}, \mathbf{H}, \vee_e) \in \mathcal{T}_{\mathbb{H}}$ and $\delta : H \to H$ w-admissible.
- The morphisms are pairs $(f, g) : (\mathbf{B_1}, \mathbf{H_1}, \vee_e^1, \delta_1) \to (\mathbf{B_2}, \mathbf{H_2}, \vee_e^2, \delta_2)$, such that:
  1. (f, g) is a good morphism pair from $(\mathbf{B_1}, \mathbf{H_1}, \vee_e^1)$ to $(\mathbf{B_2}, \mathbf{H_2}, \vee_e^2)$
  2. for all $x \in H_1$, $g(\delta_1(x)) = \delta_2(g(x))$.

The general equivalence theorem

Let $\mathbb{SBP}_{0\mathbb{H}}$ be the full subcategory of $\mathbb{SBP}_0$ consisting of algebras $\mathbf{A}$ such that $H_A \in \mathbb{H}$.

Again, we can generalize our functors and prove the following.

Theorem

Given any $\mathbb{H}$ subvariety of $\mathbb{PSH}$, $\mathbb{SBP}_{0\mathbb{H}}$ and $\mathcal{Q}_{\mathbb{H}}$ are categorically equivalent.

Hence in particular, $\mathbb{SBP}_0$ and $\mathcal{Q}_{\mathbb{PSH}}$ are categorically equivalent.

Weak Boolean product

Now we focus on the construction.

Given $\mathbf{A}$ a SBP$_0$-algebra, for each $\mathfrak{p} \in \operatorname{Max} \mathbf{B_A}$, let

$$\Theta_{\mathfrak{p}} = \{(c, c') \in \mathbf{H_A} \times \mathbf{H_A} \ s.t. \ \exists b \in \mathfrak{p} \mid \neg b \vee_e c = \neg b \vee_e c'\}.$$

Each $\Theta_{\mathfrak{p}}$ is a congruence of $\mathbf{H_A}$, moreover it holds:

Theorem

Every SBP$_0$-algebra $\mathbf{A}$ is a subdirect product of the indexed family

$$\mathbf{B_A}/\mathfrak{p} \otimes_e^{\delta} \mathbf{H_A}/\Theta_{\mathfrak{p}}$$

for some w-admissible operator δ, and for $\mathfrak{p} \in \operatorname{Max} \mathbf{B_A}$.

Weak Boolean product

Definition

A weak Boolean product of an indexed family $(\mathbf{A}_x)_{x \in X}$, $X \neq \varnothing$, of algebras is a subdirect product $\mathbf{A} \leq \prod_{x \in X} \mathbf{A_x}$, where X can be endowed with a Boolean space topology such that:

1. $[\![x = y]\!]$ is open for $x, y \in \mathbf{A}$.
2. If $x, y \in \mathbf{A}$ and N is a clopen subset of X, then $x_{|_N} \cup y_{|_{X \setminus N}} \in A$.

If $[\![x = y]\!]$ is clopen, $\mathbf{A} \leq \prod_{x \in X} \mathbf{A_x}$ is a Boolean product.

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We can prove that our construction is a weak Boolean product:

Theorem

Every SBP$_0$-algebra $\mathbf{A}$ is a weak Boolean product of the indexed family

$$\mathbf{B_A}/\mathfrak{p} \otimes_e^\delta \mathbf{H_A}/\Theta_\mathfrak{p}$$

for some w-admissible operator δ, and for $\mathfrak{p} \in \operatorname{Max} \mathbf{B_A}$.

Weak Boolean product

In particular we can exhibit the equalizer:

(1) Let $N_b = \{\mathfrak{p} \in \mathrm{Max}\,\mathbf{B_A} \mid b \in \mathfrak{p}\}$ for $b \in \mathbf{B_A}$.
Those sets are the basis of the topology.

We have that the equalizer $[\![x = y]\!]$ is open since it is equal to:

$$O = \left(N_{b_1} \cap N_{b_2} \cap \bigcup_{b \in B_1} N_b\right) \cup \left(N_{\neg b_1} \cap N_{\neg b_2} \cap \bigcup_{\neg b' \in B_2} N_{\neg b'}\right).$$

Where we have $b_1, b_2 \in \mathbf{B_A}$, $c_1, c_2 \in \mathbf{H_A}$ such that:

$x = (\neg b_1 \vee c_1) \wedge (b_1 \vee \neg c_1)$,

$y = (\neg b_2 \vee c_2) \wedge (b_2 \vee \neg c_2)$.

$B_1 = \{b \in \mathbf{B_A} \mid \neg b \vee c_1 = \neg b \vee c_2\}$

$B_2 = \{\neg b' \in \mathbf{B_A} \mid b' \vee \neg c_1 = b' \vee \neg c_2\}$.

Weak Boolean product

Notice that if $\mathbf{B_A}$ is complete,

$$\bigcup_{b \in B_1} N_b = N_{b^*} \text{ and } \bigcup_{\neg b' \in B_2} N_{\neg b'} = N_{b^{**}}$$

where $b^* = \bigvee_{b \in B_1} b$ and $b^{**} = \bigvee_{\neg b' \in B_2} \neg b'$.

Hence, $O = [\![x = y]\!]$ is clopen and we have a Boolean product.

Note: This does not characterize SBP_0-algebras with complete Boolean skeleton. Ex: it also holds in the case that $\mathbf{A}$ is the direct product of the family $\mathbf{B_A}/\mathfrak{p} \otimes_e^\delta \mathbf{H_A}/\mathfrak{p}$, for $\mathfrak{p} \in \mathrm{Max}\, \mathbf{B_A}$, where $\mathbf{B_A}$ need not be complete.

Other ideas and future work

- Generalize the hoop.

  In order to have our construction, it seems that we only need $\mathbf{H}$ to be a distributive integral lattice.

- Substitute the Boolean algebra.

  We need a distributive lattice, whose dual space is compact.
  Finite MV algebras?
  Finite Gödel algebras in order to obtain ordinal sums?

- Sheaf representation?