Kleene algebras with implication

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Kalman's functor

A *De Morgan* algebra is an algebra $\langle A, \vee, \wedge, \sim, 0, 1\rangle$ of type $(2, 2, 1, 0, 0)$ such that $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and $\sim$ satisfies

- $\sim\sim x = x$,
- $\sim(x \vee y) = \sim x \wedge \sim y$, $\sim(x \wedge y) = \sim x \vee \sim y$.

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A Kleene algebra is *centered* if it has a center. That is, an element $\mathbf{c}$ such that $\sim\mathbf{c} = \mathbf{c}$ (it is necesarily unique).

Kalman's functor

In 1958 Kalman proved that if L is a bounded distributive lattice, then

$$\mathrm{K}(L) = \{(a, b) \in L \times L : a \wedge b = 0\}$$

is a centered Kleene algebra defining

$$\begin{aligned} (a, b) \vee (d, e) &:= (a \vee d, b \wedge e), \\ (a, b) \wedge (d, e) &:= (a \wedge d, b \vee e), \\ \sim (a, b) &:= (b, a), \end{aligned}$$

$(0, 1)$ as the zero, $(1, 0)$ as the top and $(0, 0)$ as the center.

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- Kalman J.A, *Lattices with involution*. Trans. Amer. Math. Soc. 87, 485–491, 1958.

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For $(a, b) \in \mathrm{K}(L)$ we have that

$$(a, b) \wedge (0, 0) = (a \wedge 0, b \vee 0) = (0, b),$$

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Therefore, the center give us the coordinates of (a,b).

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   If $f : L \to M$ is a morphism in BDL then $\mathrm{K}(f) : \mathrm{K}(L) \to \mathrm{K}(M)$ given by $\mathrm{K}(f)(a, b) = (fa, fb)$ is a morphism of Kleene algebras.

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- Cignoli R., *The class of Kleene algebras satisfying an interpolation property and Nelson algebras*. Algebra Universalis 23, 262–292, 1986.

Kalman's functor

1. Let T be a centered Kleene algebra. Write (CK) for the following condition:

   For every x, y, if $x, y \geq \mathbf{c}$ and $x \wedge y = \mathbf{c}$ then there is z such that $z \vee \mathbf{c} = x$ and $\sim z \vee \mathbf{c} = y$.

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2. In $\mathrm{K}(L)$, if $x, y \geq \mathbf{c}$ and $x \wedge y = \mathbf{c}$ then x and y takes the form $x = (a, 0)$, $y = (b, 0)$ with $a \wedge b = 0$. In this case, $z = (a, b)$.

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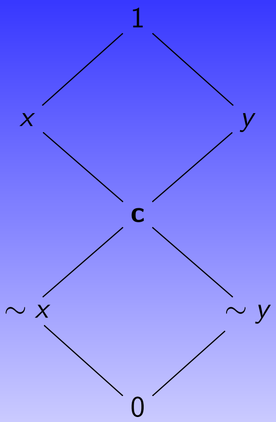
3. In an unpublished manuscript (2004) M. Sagastume proved:

   A centered Kleene algebra satisfies (IP) iff it satisfies (CK).

Centered Kleene algebra without (CK)



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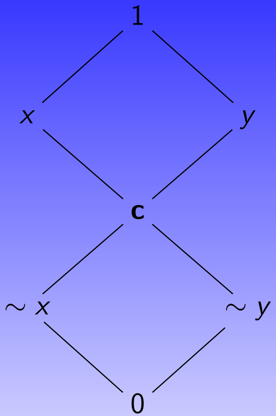

We have that $x, y \geq \mathbf{c}$ and $x \wedge y = \mathbf{c}$. However there is not z such that $z \vee \mathbf{c} = x$ and $\sim z \vee \mathbf{c} = y$.

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1. If T is a centered Kleene algebra then $\mathrm{C}(T) = \{x : x \geq \mathbf{c}\} \in \mathrm{BDL}$.
2. If $g : T \to U$ is a morphism of centered Kleene algebras then $\mathrm{C}(g) : \mathrm{C}(T) \to \mathrm{C}(U)$ given by $\mathrm{C}(g)(x) = g(x)$ is in BDL.

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3. If T is a centered Kleene algebra then $\beta : T \to \mathrm{K}(\mathrm{C}(T))$ given by $\beta(x) = (x \vee \mathbf{c}, {\sim}x \vee \mathbf{c})$ is an injective morphism of Kleene algebras. Moreover, T satisfies (CK) if and only if β is surjective.

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Theorem

There is a categorical equivalence $\mathrm{K} \dashv \mathrm{C}$ between BDL and the full subcategory of centered Kleene algebras whose objects satisfy (CK), whose unit is α and whose counit is β.

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- Sagastume, M. Categorical equivalence between centered Kleene algebras with condition (CK) and bounded distributive lattices, 2004.

Kalman's functor for Heyting algebras

1. A Nelson algebra is a Kleene algebra such that there exists
$$x \to y := x \to_{\mathrm{Hey}} (\sim x \vee y),$$
where $\to_{\mathrm{Hey}}$ is the Heyting implication,
$$(x \wedge y) \to z = x \to (y \to z).$$
2. A Nelson lattice is an involutive bounded conmutative residuated lattice which satisfies an additional equation. The varieties of Nelson algebras and Nelson lattices are term equivalent.

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Let H be a Heyting algebra where $\to$ is the Heyting implication. In $\mathrm{K}(H)$ the implication as Nelson algebra is given by

$$(a, b) \Rightarrow_{\mathrm{NA}} (d, e) = (a \to d, a \wedge e)$$

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The implication $\Rightarrow$ as Nelson lattice will be given by

$$(a, b) \Rightarrow (d, e) = ((a \rightarrow d) \wedge (e \rightarrow b), a \wedge e).$$

DLI-algebras

Definition

An algebra $(H, \wedge, \vee, \rightarrow, 0, 1)$ of type $(2, 2, 2, 0, 0)$ is a DLI-algebra if $(H, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and the following conditions are satisfied:

1. $(a \rightarrow b) \wedge (a \rightarrow d) = a \rightarrow (b \wedge d)$,
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- Celani S., *Bounded distributive lattices with fusion and implication*. Southeast Asian Bull. Math. 27, 1–10, 2003.

DLI-algebras and the functor K

We are interested in DLI-algebras in which for (a, b), (d, e) in $\mathrm{K}(H)$ is possible to define the operation

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If for instance we consider DLI-algebras with the additional condition

$$a \wedge (a \to d) \leq d$$

then we obtain that $\Rightarrow$ is a well defined map in $\mathrm{K}(H)$.

DLI^+

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Remark

Let $(H, \wedge)$ be a meet semilattice and $\to$ a binary operation on H. The following conditions are equivalent:

1. $a \wedge (a \to b) \leq b$ for every a, b.
2. For every a, b, d, if $a \leq b \to d$ then $a \wedge b \leq d$.

Kalman's functor

In the paper

- *Kleene algebras with implication* (Castiglioni, Celani and San Martín, accepted in Algebra Universalis in 2016)

we consider the category KLI whose objects are called Kleene algebras with implication: these objects are algebras $(T, \wedge, \vee, \rightarrow, \sim, \mathbf{c}, 0, 1)$ of type $(2, 2, 2, 1, 0, 0, 0)$ such that

1. $(T, \wedge, \vee, \sim, \mathbf{c}, 0, 1)$ is a centered Kleene algebra,
2. $(T, \wedge, \vee, \rightarrow, 0, 1)$ is a DLI-algebra.
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Theorem

There is a categorical equivalence $\mathrm{K} \dashv \mathrm{C}$ between DLI^{+} and the full subcategory of KLI whose objects satisfy (CK), whose unit is α and whose counit is β.

Final remarks

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- If $H \in \mathrm{DLI}^+$ then $\mathrm{K}(H)$ is a DLI-algebra.
- If $H \in \mathrm{DLI}^+$ then the implication in $\mathrm{K}(H)$ is interdefinable with other operation, and $\mathrm{K}(H)$ with this operation is an algebra with fusion.
- This construction also generalizes some given for the case of integral commutative residuated lattices with bottom.