Fuzzy neighbourhood semantics

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Outline

Motivations

Most relevant models of reasoning with imperfect information try to address and formalize two different central notions:

- **Fuzziness / Graduality**: formalized by (truth-functional) many-valued models, $w : \mathcal{L} \mapsto [0,1]$.
  $w(\varphi), w(\psi) \in [0,1]$; $w(\varphi \rightarrow \psi) = 1$ iff $w(\varphi) \leq w(\psi)$
- **Uncertainty**: formalized by modal structures, $\mathcal{M} = \langle W, \mu \rangle$ where W is a non empty set of possible worlds and $\mu : W \mapsto [0,1]$ uncertainty distribution (e.g. probability)
  $\mathcal{M} \models Pr^{\alpha}\varphi$ iff $\sum_{w \models \varphi} \mu(w) \geq \alpha$

Fuzzy Logics

Our aim is to study modal expansions of systems of fuzzy logics (in the sense of HÁJEK), with the following characteristics:

Language(non modal):

$$\varphi ::= p \mid \bot \mid \top \mid \varphi_0 \wedge \varphi_1 \mid \varphi_0 \vee \varphi_1 \mid \varphi_0 \& \varphi_1 \mid \varphi_0 \rightarrow \varphi_1$$

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Logic associated to a class of RLs (typically FL$_{ew}$ algebras):

- i.e. (complete) residuated lattices (**RLs**) $\mathbf{A} = \langle A, 0, 1, \wedge, \vee, *, \Rightarrow \rangle$ as set of truth-values.
- The logic associated with a class of **RLs**, $\mathcal{C}$, will be denoted by

$$\Lambda(\mathcal{C}) = \bigcap_{\mathbf{A} \in \mathcal{C}} \{\varphi \mid \models_{\mathbf{A}} \varphi = 1\}$$

Fuzzy Modal Logics

Our aim is to expand an axiomatization of the non-modal logic $\Lambda(\mathcal{C})$ into one of the modal logic.

Definition

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Definition (RL with unary operators $\Box$ and $\Diamond$)

A modal algebra is an algebra $M\mathbf{A} = \langle \mathbf{A}, \Box, \Diamond \rangle$ where $\mathbf{A}$ is a **RL** and $\Box$, $\Diamond$ are unary operations satisfying:

$$\text{If } \varphi \Leftrightarrow \psi = 1 \text{ then } \Box\varphi \Leftrightarrow \Box\psi = 1 \text{ and } \Diamond\varphi \Leftrightarrow \Diamond\psi = 1$$

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Given a class of **RLs**, $\mathcal{C}$, we denote with $\Lambda(M\mathcal{C})$ the modal logic associated with the class of $M\mathcal{C}$-algebras : $\{\langle \mathbf{A}, \Box, \Diamond \rangle | \mathbf{A} \in \mathbf{RLs}\}$

Fuzzy Modal Logics (2)

For any axiomatic of a class of **RLs**, $\mathcal{C}$, we define the fuzzy modal logic $E(\mathcal{C})$ as the logic axiomatized by that axiomatic extended with the following rules:

$$\frac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi} \qquad \frac{\varphi \leftrightarrow \psi}{\Diamond\varphi \leftrightarrow \Diamond\psi}$$

For each class of **RLs**, $\mathcal{C}$, the logic obtained in this way will be called *$\mathcal{C}$-classical modal logic*.
Clearly, $E(\mathcal{C})$ is algebraizable, since the new rules ensure the congruence for the new operators $\Box$ and $\Diamond$.

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That is essentially the algebraic counterpart of the approach introduced in the previous talk by Cintula and Noguera.

Algebraic Completeness

In that context, it is possible to prove the following:

Theorem

Let $\mathcal{C}$ be a a class of **RLs**. *Then:*

1. $\Vdash_{E(\mathcal{C})} \varphi$ *iff* φ *is valid in all* $M\mathcal{C}$*- algebras.*
2. *For any set* T *of formulae, we have:*

$$T \Vdash_{E(\mathcal{C})} \varphi \text{ iff } \quad \forall \mathbf{A} \in \mathcal{C}, \forall e : \mathcal{L}_{\Box,\Diamond} \mapsto \mathbf{A} : \\ \text{if } e(T) = 1 \ \text{ then } e(\varphi) = 1$$

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Remark: The consequence relation mentioned in 2 is *global.*
In particular, according to that definition: $\varphi \leftrightarrow 1 \Vdash_{E(\mathcal{C})} \Box\varphi \leftrightarrow \Box 1$.

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For these both reasons, we are going to consider an alternative semantics: **Neighborhood**.

Neighborhood Semantics (1)

Assume we have fixed a complete residuated lattice:

$$\mathbf{A} = \langle A, 0, 1, \wedge, \vee, *, \Rightarrow \rangle$$

as set of truth-values.

Definition

$\mathbf{A}$-valued Neighborhood frame: $\langle W, N^{\Box}, N^{\Diamond} \rangle$ such that

- W is a nonempty set
- $N^{\Box}, N^{\Diamond} : \mathcal{F}(W) \times W \to A$
  where $\mathcal{F}(W)$ is a (suitable) subalgebra of $\mathbf{A}^{\mathbf{W}}$

Neighborhood Semantics (2)

Definition

$\mathbf{A}$-valued Neighborhood model: $\langle W, N^{\Box}, N^{\Diamond}, e\rangle$, i.e. a frame plus an evaluation $e : Var \times W \longrightarrow \mathbf{A}$, that is extended to formulas under the following conditions:

- $e(\cdot, w)$ is, for all $w \in W$, an algebraic homomorphism for the connectives in the algebraic signature of $\mathbf{A}$,
- $e(\Box\varphi, w) = N^{\Box}(\mu_\varphi, w)$.
- $e(\Diamond\varphi, w) = N^{\Diamond}(\mu_\varphi, w)$.

where $\mu_\varphi = e(\varphi, \cdot)$

The values $N^{\Box}(\mu_\varphi, w)$ and $N^{\Diamond}(\mu_\varphi, w)$ are meant as the necessity and possibility degrees of φ at w, resp.

Validity

Definition

A formula φ is true

- at a world w in a model $\mathcal{N}$, written $(\mathcal{N}, w) \models \varphi$, if $e(\varphi, w) = 1$

A formula φ is valid:

- in a model $\mathcal{N}$, written $\mathcal{N} \models \varphi$, iff for every w in $\mathcal{N}$, $(\mathcal{N}, w) \models \varphi$.
- in a class of models N, written $\models_{\mathsf{N}} \varphi$, if for any $\mathcal{N} \in \mathsf{N}$, $\mathcal{N} \models \varphi$

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For any class of $\mathbf{A}$-valued neighborhood models N:

If $\models_{\mathsf{N}} \varphi \leftrightarrow \psi$ then $\models_{\mathsf{N}} \Box\varphi \leftrightarrow \Box\psi$ and $\models_{\mathsf{N}} \Diamond\varphi \leftrightarrow \Diamond\psi$.

The $E(\mathcal{C})$ logic: completeness

By an adaptation of the standard technique of the canonical model construction, one can prove:

Theorem

Let $\mathcal{C}$ be a strongly complete axiomatizable class of Residuated lattices. Then the logic $E(\mathcal{C})$ is sound and weak complete with respect to the class of $\mathcal{C}$- neighborhood models.

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Local consequence relation:

$$\Gamma \vdash_{E(\mathcal{C})} \varphi \text{ iff } \forall \mathcal{N}\ \forall \omega \in W: \text{ if } (\mathcal{N}, \omega) \models \Gamma \text{ then } (\mathcal{N}, \omega) \models \varphi$$

Correspondence

Many of the following usual schemes are not valid in the class of all neighborhood models, e.g.:

$$\begin{array}{ll} \mathbf{M}^{\wedge}_{\Box} & \Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi) \\ \mathbf{C}^{\vee}_{\Diamond} & (\Diamond\varphi \vee \Diamond\psi) \leftrightarrow \Diamond(\varphi \vee \psi) \\ \mathbf{N} & \Box\top \end{array}$$

But we can define subclasses of neighborhood frames validating them by requiring conditions over the functions $N^{\Box}$ and $N^{\Diamond}$, e.g.:

$$\begin{array}{rrcl} (\mathbf{e}^{\wedge}_{\Box}) & N^{\Box}(f \wedge g, w) & = & N^{\Box}(f, w) \wedge N^{\Box}(g, w). \\ (\mathbf{e}^{\vee}_{\Diamond}) & N^{\Diamond}(f \vee g, w) & = & N^{\Diamond}(f, w) \vee N^{\Diamond}(g, w). \\ (\mathbf{n}) & N^{\Box}(\overline{1}, w) = 1 \ . & & \end{array}$$

In previous works, we have shown how neighborhoods semantics is useful to define fuzzy versions of two well-known belief modal logics:

(i) $KD45(\mathcal{C})$: a $KD45$-like modal extension of the logic $\mathcal{C}$ of a given class of complete residuated lattice $\mathbf{A}$, assuming that $\mathcal{C}$ is strongly complete[1]

(ii) $Prob(Ł_n)$: a probabilistic-like modal logic over $Ł_n$, the $(n+1)$-valued Łukasiewicz logic, with values $\{0, 1/n, \ldots, (n-1)/n, 1\}$

[1] For instance, Gödel $[0, 1]$-valued logic or finitely-valued Łukasiewicz logic.

The $KD45(\mathcal{C})$ logic: axiomatic definition

Axioms and rules of $KD45(\mathcal{C})$ are those of $\mathcal{C}$ plus:

Axioms

I : $\Box\neg\varphi \leftrightarrow \neg\Diamond\varphi$

$E^{\wedge}_{\Box}$: $\Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$

$E^{\vee}_{\Diamond}$: $\Diamond(\varphi \vee \psi) \leftrightarrow (\Diamond\varphi \vee \Diamond\psi)$

N : $\Box\top$

D : $\Box\varphi \to \Diamond\varphi$

$4_{\Box}$: $\Box\varphi \to \Box\Box\varphi$

$4_{\Diamond}$: $\Diamond\Diamond\varphi \to \Diamond\varphi$

$5_{\Box}$: $\Diamond\Box\varphi \to \Box\varphi$

$5_{\Diamond}$: $\Diamond\varphi \to \Box\Diamond\varphi$

Rules

$RE_{\Box}$: From $\varphi \leftrightarrow \psi$ infer $\Box\varphi \leftrightarrow \Box\psi$

$RE_{\Diamond}$: From $\varphi \leftrightarrow \psi$ infer $\Diamond\varphi \leftrightarrow \Diamond\psi$

The $KD45(\mathcal{C})$ logic: neighborhood semantics

$\mathbf{A}$-valued belief neighborhood models are models ($\mathbf{A} \in \mathcal{C}$) $\mathcal{N} = \langle W, N^{\Box}, N^{\Diamond}, e \rangle$ such that, for every world w and formulas φ and ψ, the following conditions hold:

$$
\begin{array}{rl}
(\mathbf{e}^{\wedge}_{\Box}) & N^{\Box}(\mu_\varphi \wedge \mu_\psi, w) \;=\; N^{\Box}(\mu_\varphi, w) \wedge N^{\Box}(\mu_\psi, w) \\
(\mathbf{e}^{\vee}_{\Diamond}) & N^{\Diamond}(\mu_\varphi \vee \mu_\psi, w) \;=\; N^{\Diamond}(\mu_\varphi, w) \vee N^{\Diamond}(\mu_\psi, w) \\
(\mathbf{n}) & N^{\Box}(\overline{1}, w) = 1 \\
(\mathbf{d}) & N^{\Box}(\mu_\varphi, w) \;\leq\; N^{\Diamond}(\mu_\varphi, w) \\
(\mathbf{iv}_{\Box}) & N^{\Box}(\mu_\varphi, w) \;\leq\; N^{\Box}(\mu_{\Box\varphi}, w) \\
(\mathbf{iv}_{\Diamond}) & N^{\Diamond}(\mu_{\Diamond\varphi}, w) \;\leq\; N^{\Diamond}(\mu_\varphi, w) \\
(\mathbf{v}_{\Box}) & N^{\Diamond}(\mu_{\Box\varphi}, w) \;\leq\; N^{\Box}(\mu_\varphi, w) \\
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By an adaptation of the standard technique of the canonical model construction, one can prove:

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Open question: is there a smaller, simpler class of $\mathbf{A}$-valued belief neighborhood models while keeping completeness?

The $KD45(\mathcal{C})$ logic: connections to possibilistic models?

Possibilistic models are well-known graded, but qualitative, models of belief on classical propositions (Dubois-Prade), extending KD45. Several extensions proposed for fuzzy propositions.

- Given a possibility distribution $\pi : W \to \mathbf{A}$, define the **A**-valued possibilistic model $\mathcal{N} = \langle W, N_\pi^\Box, N_\pi^\Diamond, e\rangle$, where

$$N_\pi^\Box(\mu_\varphi, w) = \inf_{w' \in W} [\pi(w') \Rightarrow \mu_\varphi(w')]$$
$$N_\pi^\Diamond(\mu_\varphi, w) = \sup_{w' \in W} [\pi(w') \odot \mu_\varphi(w')]$$

- Note that $N_\pi^\Box$ and $N_\pi^\Diamond$ are independent of local world w.
- The class of possibilistic models is a subclass of the belief models.
- Axiomatize them is an open problem (known for $\mathcal{A} =$ Gödel logic)

$Prob(Ł_n)$: axiomatics

The axioms and rules of $Prob(Ł_n)$ are those of $Ł_n$ plus the following:

Axioms

I : $\Box\neg\varphi \equiv \neg\Box\varphi$

Ad : $\Box(\varphi \oplus \psi) \equiv \Box\varphi \oplus (\Box\psi \odot \neg\Box(\varphi \odot \psi))$

N : $\Box\top$

$4_\Box$: $\Box\varphi \equiv \Box\Box\varphi$

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Axiom (I) captures the self-duality property of states while axiom (Ad) the finite additivity of states. Axiom (K) is derivable.

$Prob(Ł_n)$: semantics

Intended semantics of $\Box\varphi$: φ is probable, truth-degree$(\Box\varphi)$ = probability of φ.

An $Ł_n$-valued probabilistic neighborhood model is a triple $\mathcal{N} = \langle W, \Pi, e\rangle$, where:

- W is a nonempty set worlds
- $\Pi : (Ł_n)^W \times W \mapsto Ł_n$, such that $\Pi(\cdot, w)$ is a state:
  - $\Pi(\overline{1}, w) = 1$
  - $\Pi(\mu_\varphi \oplus \mu_\psi, w) = \Pi(\mu_\varphi, w) + \Pi(\mu_\psi, w)$, if $\mu_\varphi \odot \mu_\psi = \overline{0}$
- $\bar{e}(\cdot, w)$ is an algebraic homomorphism for the connectives in the algebraic signature of $\mathbf{Ł_n}$,
- $\bar{e}(\Box\varphi, w) = \Pi(\mu_\varphi, w)$

$Prob(Ł_n)$: completeness

Again, based on the canonical model construction, we can formulate a completeness result for $Prob(Ł_n)$:

Theorem

The logic $Prob(Ł_n)$ is sound and weak complete with respect to the class of $Ł_\mathbf{n}$-valued neighborhood probabilistic models.

Conclusions and future work

We have introduced a many-valued variant of the classical neighborhood semantics.

Some relevant open questions:

- A crucial assumption in $E(\mathcal{C})$ is that underlying logic $\mathcal{C}$ is strongly complete, ruling out a number of the most well-known fuzzy logics. Is it possible to relax this assumption?
- To reason about numerical degrees of uncertainty, one would need to introduce them as truth constants in the modal language.
- Decidability and complexity issues.

THANK YOU !