An Abstract Approach to Consequence Relations

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Tarskian consequence

A *Tarskian consequence relation (tcr)* on $\mathcal{L}$-formulas is a relation $\vdash \subseteq \wp(Fm_{\mathcal{L}}) \times Fm_{\mathcal{L}}$ such that for all $\Gamma \cup \Delta \cup \{\varphi, \psi\} \subseteq Fm_{\mathcal{L}}$:

1. $\Gamma \vdash \varphi$ whenever $\varphi \in \Gamma$ *(Reflexivity)*
2. If $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \varphi$ *(Monotonicity)*
3. If $\Delta \vdash \psi$ and $\Gamma \vdash \varphi$ for every $\varphi \in \Delta$, then $\Gamma \vdash \psi$ *(Cut)*

A tcr is *substitution-invariant* if $\Gamma \vdash \varphi$ implies $\sigma(\Gamma) \vdash \sigma(\varphi)$ for all $\mathcal{L}$-substitutions σ ($\sigma(\Gamma)$ defined pointwise).

Blok-Jónsson consequence: The vanilla theory

An *abstract consequence relation (acr)* over the set X is a relation $\vdash \subseteq \wp(X) \times X$ such that for all $\Gamma \cup \Delta \cup \{a\} \subseteq X$:

1. $\Gamma \vdash a$ whenever $a \in \Gamma$ *(Reflexivity)*
2. If $\Gamma \vdash a$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash a$ *(Monotonicity)*
3. If $\Delta \vdash a$ and $\Gamma \vdash b$ for every $b \in \Delta$, then $\Gamma \vdash a$ *(Cut)*

Similarity of acr's

Acr's $\vdash_1$ and $\vdash_2$ over X_1 and X_2 resp. are *similar* if there are mappings

$$\tau \colon X_1 \to \wp(X_2) \qquad \rho \colon X_2 \to \wp(X_1)$$

such that for every $\Gamma \cup \{a\} \subseteq X_1$ and every $\Delta \cup \{b\} \subseteq X_2$:

S1	$\Gamma \vdash_1 a$ iff $\tau(\Gamma) \vdash_2 \tau(a)$	S3	$a \dashv\vdash_1 \rho(\tau(a))$
S2	$\Delta \vdash_2 b$ iff $\rho(\Delta) \vdash_1 \rho(b)$	S4	$b \dashv\vdash_2 \tau(\rho(b))$

Put differently, the acr's $\vdash_1$ and $\vdash_2$ are similar when:

- $\vdash_1$ is faithfully translatable via the mapping τ into $\vdash_2$ (S1)
- $\vdash_2$ is faithfully translatable via the mapping ρ into $\vdash_1$ (S2)
- the two mappings ρ and τ are mutually inverse (S3 and S4)

Examples of similarities

- Algebraisability (similarity between a tcr and the equational consequence relation of some class of algebras);
- Gentzenisability (similarity between a tcr and some consequence relations on sequents);
- Same-environment similarities (e.g. algebraisable tcr's that have the same equivalent algebraic semantics with different transformers).

Limits of the vanilla theory

The set X is a "black box": it carries no inner structure, whence e.g. we can give no notion of endomorphism other than the trivial one (a permutation). Substitution-invariance cannot simply be expressed.

With respect to their Tarskian competitor, Blok and Jónsson have attained a greater level of generality at the expense of the *applicability* of the theory (Hilbert systems, matrices, etc.)

Action-invariant acr's

The monoid $\mathbf{M} = (M, \circ, 1)$ is said to *act* on non-empty set X if there is an operation $\cdot : M \times X \to X$ such that, for all $\sigma, \sigma' \in M$ and all $a \in X$:

$$(\sigma \circ \sigma') \cdot a = \sigma \cdot (\sigma' \cdot a) .$$

The operation $\cdot$ is called *scalar product*, and the scalars in M are called *actions*. We write $\sigma(a)$ instead of $\sigma \cdot a$.

When $\mathbf{M}$ acts on X, an acr $\vdash$ on X is called *action-invariant* if, for any $\sigma \in M$, for any $\Gamma \subseteq X$ and for any $a \in X$,

$$\text{if } \Gamma \vdash a, \text{ then } \sigma(\Gamma) \vdash \sigma(a) .$$

The general theory (BJ, Galatos-Tsinakis)

- Consider symmetric (multiple-conclusion) versions of the acr's;
- "Lift" the actions and the transformers to the level of *powersets*;
- $\wp(M)$ is the universe a complete residuated lattice, with complex product as the residuated operation (the *scalars*); $\wp(X)$ is the universe of a complete lattice (the *vectors*); Scalar product is a biresiduated map that satisfies the usual properties of a monoid action.
- Go fully abstract: acr's on complete lattices as *preorders* on complete lattices that contain the converse of the lattice order.
- Abstractly, equivalence of such acr's can be defined by tweaking similarity in such a way as to accommodate action-invariance.

Limits of the general theory

The idea of a consequence relation as a preorder on a complete lattice that contains the converse of the lattice order is not general enough: it rules out important cases where we have non-idempotent operations of premiss and conclusion aggregation.

Example: multiset consequence (internal consequence relations of substructural sequent calculi, resource-conscious versions of logics from commutative integral residuated lattices, etc.) can be *only* treated as consequence relation on *sequents* but not as consequence relation on *formulas*

So we could use the theory of algebraization of Gentzen systems but this would add an unnecessary level of complexity . . .

Deductive relations

Definition

A *deductive relation* (dr) $\vdash$ on a dually integral Abelian po-monoid $\mathbf{R} = \langle R, \leq, +, 0 \rangle$ is a preorder on R such that for every $a, b, c \in R$:

1. If $a \leq b$, then $b \vdash a$.
2. If $a \vdash b$, then $a + c \vdash b + c$.

Examples (1)

Example (Tarski)

Any tcr $\vdash$ on the language $\mathcal{L}$ canonically gives rise to a dr on the Abelian po-monoid

$$\mathbf{R} = \langle \wp\left(Fm_{\mathcal{L}}\right), \subseteq, \cup, \emptyset \rangle .$$

Example (Blok–Jónsson)

Any acr $\vdash$ over the set X canonically gives rise to a dr on the Abelian po-monoid

$$\mathbf{R} = \langle \wp\left(X\right), \subseteq, \cup, \emptyset \rangle .$$

Examples (2)

Example (Multiset consequence)

Let $\mathcal{L}$ be a language, and let $Fm^{\flat}_{\mathcal{L}}$ be the set of finite multisets of $\mathcal{L}$-formulas. A *multiset deductive relation* (mdr) on $\mathcal{L}$ is a preorder $\vdash$ on $Fm^{\flat}_{\mathcal{L}}$ that satisfies the following additional postulates:

1. If $\ulcorner\varphi_1, \ldots, \varphi_n\urcorner \leq \ulcorner\psi_1, \ldots, \psi_m\urcorner$, then $\ulcorner\psi_1, \ldots, \psi_m\urcorner \vdash \ulcorner\varphi_1, \ldots, \varphi_n\urcorner$.
2. If $\ulcorner\psi_1, \ldots, \psi_m\urcorner \vdash \ulcorner\varphi_1, \ldots, \varphi_n\urcorner$, then

$$\ulcorner\gamma_1, \ldots, \gamma_m\urcorner \uplus \ulcorner\psi_1, \ldots, \psi_m\urcorner \vdash \ulcorner\gamma_1, \ldots, \gamma_m\urcorner \uplus \ulcorner\varphi_1, \ldots, \varphi_n\urcorner.$$

So, any mdr $\vdash$ on the language $\mathcal{L}$ is a dr on

$$\mathbf{R} = \left\langle Fm^{\flat}_{\mathcal{L}}, \leq, \uplus, \varnothing \right\rangle.$$

$(\mathfrak{X} \uplus \mathfrak{Y}(\varphi) = \mathfrak{X}(\varphi) + \mathfrak{Y}(\varphi)$; $\mathfrak{X} \leq \mathfrak{Y}$ iff for all φ, $\mathfrak{X}(\varphi) \leq \mathfrak{Y}(\varphi))$.

Examples (2)

Example (Fuzzy consequence)

Let $Fm_{\mathcal{L}}$ be the set of formulas of Pavelka's logic $\vdash^{\mathrm{Evl}}$ (a.k.a. logic with evaluated syntax). Then the relation $\vdash$ on fuzzy sets of formulas defined as:

$$\Gamma \vdash \Delta \qquad \text{iff for each } \varphi \text{ we have: } \Gamma \vdash^{\mathrm{Evl}}_{\alpha} \langle \varphi, \beta \rangle \text{ and } \Delta(\varphi) = \alpha \otimes \beta$$

is a dr over

$$\mathbf{R} = \left\langle [0,1]^{Fm_{\mathcal{L}}}, \leq, \vee, \emptyset \right\rangle.$$

where $\emptyset(\varphi) = 0$ and $\vee$ is pointwise supremum.

Deductive operators

Definition

A *deductive operator* (do) on a dually integral Abelian po-monoid $\mathbf{R} = \langle R, \leq, +, 0 \rangle$ is a map $\delta \colon R \to \mathcal{P}(R)$ such that for every $a, b, c \in R$:

1. $a \in \delta(a)$.
2. If $a \leq b$, then $\delta(a) \subseteq \delta(b)$.
3. If $a \in \delta(b)$, then $\delta(a) \subseteq \delta(b)$.
4. If $a \in \delta(b)$, then $a + c \in \delta(b + c)$.

Deductive systems

Definition

A *deductive system* (ds) on a dually integral Abelian po-monoid $\mathbf{R} = \langle R, \leq, +, 0 \rangle$ is a family $\{X_a : a \in R\} \subseteq \mathcal{P}(R)$ of down-sets of $\langle R, \leq \rangle$ such that for every $a, b, c \in R$:

1. $a \in X_b$ if and only if $X_a \subseteq X_b$.
2. If $X_a \subseteq X_b$, then $X_{a+c} \subseteq X_{b+c}$.

Lattices of drs, dos, dss

Given a dually integral Abelian po-monoid $\mathbf{R} = \langle R, \leq, +, 0\rangle$, we denote by $Rel(\mathbf{R})$, $Oper(\mathbf{R})$ and $Sys(\mathbf{R})$ the sets of drs, dos, and dss on $\mathbf{R}$, respectively.

The structures $\langle Rel(\mathbf{R}), \subseteq\rangle$, $\langle Oper(\mathbf{R}), \preccurlyeq\rangle$ and $\langle Sys(\mathbf{R}), \lessdot\rangle$, where

$$\delta \preccurlyeq \gamma \Longleftrightarrow \delta(a) \subseteq \gamma(a) \text{ for every } a \in R$$
$$\{X_a : a \in R\} \lessdot \{Y_a : a \in R\} \Longleftrightarrow X_a \subseteq Y_a \text{ for every } a \in R,$$

are complete lattices.

The Trinity

Theorem

If $\mathbf{R} = \langle R, \leq, +, 0\rangle$ *is a dually integral Abelian po-monoid, then the lattices* $\langle Rel(\mathbf{R}), \subseteq\rangle$, $\langle Oper(\mathbf{R}), \preccurlyeq\rangle$ *and* $\langle Sys(\mathbf{R}), \lessdot\rangle$ *are isomorphic.*

The isomorphisms are implemented by the maps $f\colon Oper(\mathbf{R}) \to Sys(\mathbf{R})$ and $g\colon Oper(\mathbf{R}) \to Rel(\mathbf{R})$ defined by:

$$f(\delta) = \{\delta(a) : a \in R\};$$
$$g(\delta) = \{\langle a, b\rangle : b \in \delta(a)\}.$$

Partially ordered semirings

Definition

A *partially ordered semiring* (po-semiring) is a structure $\mathbf{A} = \langle A, \leq, +, \cdot, 0, 1 \rangle$ where:

1. $\langle A, \cdot, 1 \rangle$ is a monoid;
2. $\langle A, \leq, +, 0 \rangle$ is an Abelian po-monoid;
3. $\sigma \cdot 0 = 0 \cdot \sigma = 0$ for all $\sigma \in A$;
4. for every $\sigma, \pi, \varepsilon \in A$ we have

$$\pi \cdot (\sigma + \varepsilon) = (\pi \cdot \sigma) + (\pi \cdot \varepsilon) \text{ and } (\sigma + \varepsilon) \cdot \pi = (\sigma \cdot \pi) + (\varepsilon \cdot \pi).$$

5. if $\sigma \leq \pi$ and $0 \leq \varepsilon$, then $\sigma \cdot \varepsilon \leq \pi \cdot \varepsilon$ and $\varepsilon \cdot \sigma \leq \varepsilon \cdot \pi$.

A po-semiring $\mathbf{A} = \langle A, \leq, +, \cdot, 0, 1 \rangle$ is dually integral iff $\langle A, \leq, +, 0 \rangle$ is dually integral as a po-monoid.

Po-semirings of substitutions

Example

Let Subst($\mathbf{Fm}_\mathcal{L}$) be the set of *substitutions* of $\mathbf{Fm}_\mathcal{L}$. The structure

$$\boldsymbol{\Sigma} = \langle \mathrm{Subst}(\mathbf{Fm}_\mathcal{L})^\flat, \leq, \uplus, \cdot, 0, 1 \rangle,$$

where, for $\mathfrak{X} = \ulcorner \sigma_1, \ldots, \sigma_n \urcorner$, $\mathfrak{Y} = \ulcorner \pi_1, \ldots, \pi_m \urcorner$, $\sigma \in \mathrm{Subst}(Fm_\mathcal{L})$,

$$\mathfrak{X} \cdot \mathfrak{Y} = \ulcorner \sigma_1 \circ \pi_1, \ldots, \sigma_1 \circ \pi_m, \ldots, \sigma_n \circ \pi_1, \ldots, \sigma_n \circ \pi_m \urcorner,$$

$$1(\sigma) = \begin{cases} 1, \text{ if } \sigma = id_{Fm_\mathcal{L}} \\ 0, \text{ otherwise,} \end{cases}$$

$$0(\sigma) = 0,$$

is a dually integral po-semiring.

Modules over po-semirings

Definition

Let **A** be a dually integral po-semiring. An **A**-*module* is a structure $\mathbf{R} = \langle R, \leq, +, 0, * \rangle$ where $\langle R, \leq, +, 0 \rangle$ is a dually integral Abelian po-monoid and $*\colon A \times R \to R$ is a map that is order-preserving in both coordinates, and s.t.

1. $(\sigma \cdot \pi) * a = \sigma * (\pi * a)$;
2. $1 * a = a$;
3. $0^{\mathbf{A}} * a = 0^{\mathbf{R}}$;
4. $(\sigma * a) +^{\mathbf{R}} (\sigma * b) = \sigma * (a +^{\mathbf{R}} b)$;
5. $(\sigma +^{\mathbf{A}} \pi) * a = (\sigma * a) +^{\mathbf{R}} (\pi * a)$.

Modules of substitutions

Example

Consider

$$\mathbf{\Sigma} = \langle \mathrm{Subst}(\mathbf{Fm}_{\mathcal{L}})^{\flat}, \leq, \uplus, \cdot, 0, 1 \rangle,$$

and let $\mathbf{R} = \left\langle Fm^{\flat}_{\mathcal{L}}, \leq, \uplus, \emptyset, * \right\rangle$, where for

$$\sigma = \ulcorner \sigma_1, \ldots, \sigma_n \urcorner \text{ and } \varphi = \ulcorner \varphi_1, \ldots, \varphi_m \urcorner.$$

we set

$$\sigma * \varphi = \ulcorner \sigma_1(\varphi_1), \ldots, \sigma_1(\varphi_m), \ldots, \sigma_n(\varphi_1), \ldots, \sigma_n(\varphi_m) \urcorner.$$

$\mathbf{R}$ is a $\mathbf{\Sigma}$-module.

Action-invariant dos on modules over po-semirings

Definition

An *action-invariant deductive operator* on an **A**-module $\mathbf{R} = \langle R, \leq, +, 0, * \rangle$ is a deductive operator δ on $\langle R, \leq, +, 0 \rangle$ such that for every $\sigma \in A$ and $a, b \in R$:

$$\text{if } a \in \delta(b), \text{ then } \sigma * a \in \delta(\sigma * b).$$

The category of A-modules

$\mathbf{A}$-Md is the category whose objects are $\mathbf{A}$-modules and whose arrows are po-monoid homomorphisms τ that respect the monoidal action:

$$\tau(\sigma * a) = \sigma * \tau(a) \text{ for every } \sigma \in A \text{ and } a \in R.$$

Theories

Lemma

Let δ be an action-invariant do on the $\mathbf{A}$-module $\mathbf{R}$. The structure

$$\mathbf{R}_\delta = \langle \delta[R], \subseteq, +^\delta, \delta(0), *^\delta \rangle$$

where $\delta(a) +^\delta \delta(b) = \delta(a+b)$ and $\sigma *^\delta \delta(a) = \delta(\sigma * a)$, is an object of $\mathbf{A}$-Md and the map $\delta\colon \mathbf{R} \to \mathbf{R}_\delta$ is an arrow of $\mathbf{A}$-Md.

Structural representations

Definition

Let δ and γ be two action-invariant dos on the **A**-modules **R** and **S**, respectively. A *structural representation* of δ into γ is an injective morphism $\Phi\colon \mathbf{R}_\delta \longrightarrow \mathbf{S}_\gamma$ that reflects the order.

The structural representation $\Phi\colon \mathbf{R}_\delta \longrightarrow \mathbf{S}_\gamma$ is said to be *induced* if there is a morphism $\tau\colon \mathbf{R} \to \mathbf{S}$ that makes the following diagram commute:

$$\begin{array}{ccc} \mathbf{R} & \xrightarrow{\ \tau\ } & \mathbf{S} \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \gamma} \\ \mathbf{R}_\delta & \xrightarrow{\ \Phi\ } & \mathbf{S}_\gamma \end{array}$$

Projective A-modules

Definition

An $\mathbf{A}$-module $\mathbf{R}$ has the *representation property* (REP) if for any $\mathbf{A}$-module $\mathbf{S}$ and action-invariant dos δ and γ on $\mathbf{R}$ and $\mathbf{S}$ respectively, every structural representation of δ into γ is induced.

Definition

An object $\mathbf{R}$ in $\mathbf{A}$-Md is *onto-projective* if for every pair of morphisms $f\colon \mathbf{S} \to \mathbf{T}$ and $g\colon \mathbf{R} \to \mathbf{T}$ between $\mathbf{A}$-modules with f onto, there is a morphism $h\colon \mathbf{R} \to \mathbf{S}$ such that $f \circ h = g$.

Theorem

An $\mathbf{A}$-module has the REP iff it is onto-projective in $\mathbf{A}$-Md.

Characterisation of cyclic and projective A-modules

Definition

An **A**-module **R** is *cyclic* if there is $v \in R$ such that $R = \{\sigma * v : \sigma \in A\}$.

Theorem

Let **R** *be an* **A***-module. The following conditions are equivalent:*

1. **R** *is cyclic and onto-projective.*
2. *There is a retraction* $f : \mathbf{A} \to \mathbf{R}$.
3. *There are* $\mu \in A$ *and* $v \in R$ *such that* $\mu * v = v$ *and* $A * \{v\} = R$ *and for every* $\sigma, \pi \in A$:

$$\text{if } \sigma * v \leq \pi * v, \text{ then } \sigma \cdot \mu \leq \pi \cdot \mu.$$

The motivating example

Theorem

The $\mathbf{\Sigma}$*-module*

$$\mathbf{R} = \left\langle Fm_{\mathcal{L}}^{\flat}, \uplus, \emptyset, \leq, * \right\rangle$$

of finite multisets of formulas of a sentential language is cyclic and onto-projective. In particular, this implies that it has the REP.

Thank you...

...for your attention!