Adjunctions as translations between relative equational consequences

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- Twist constructions:

$$\text{Distributive lattices} \longmapsto \text{Kleene lattices}$$

$$\text{Lattices} \longmapsto \text{Bilattices}$$

Adjoint Functors

Definition

A pair of functors $\mathcal{F}\colon \mathsf{X} \longleftrightarrow \mathsf{Y}\colon \mathcal{G}$ is an adjunction if there is a pair of natural transformation $\eta\colon 1_{\mathsf{X}} \to \mathcal{G}\mathcal{F}$ and $\epsilon\colon \mathcal{F}\mathcal{G} \to 1_{\mathsf{Y}}$ such that

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  right adjoints = generalized twist constructions.

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Twist constructions

Well-known example

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- Do the κ-power of A for some cardinal κ. (above $\kappa = 2$).
- Select in some elements $G(A) \subseteq A^\kappa$ and define new basic operations for $G(\boldsymbol{A})$ which are κ-sequences of operations of $\boldsymbol{A}$.

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The κ-th matrix power of X is the class

$$\mathsf{X}^{[\kappa]} := \mathbb{I}\{\boldsymbol{A}^{[\kappa]} : \boldsymbol{A} \in \mathsf{X}\}.$$

Compatible Equations

Definition

Let X be a class of algebras of language $\mathscr{L}_X$ and $\mathscr{L} \subseteq \mathscr{L}_X$. A set of equations θ in one variable is compatible with $\mathscr{L}$ in X if for every n-ary operation $\varphi \in \mathscr{L}$ we have that:

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- We obtain a functor

$$\theta_{\mathscr{L}} : \mathsf{X} \to \mathbb{I}\{\boldsymbol{A}(\theta, \mathscr{L}) : \boldsymbol{A} \in \mathsf{X}\}.$$

Generalized twist constructions

- According to the previous abstractions, a generalized twist construction between two quasi-varieties K and V is a functor of the form
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2. $\theta_{\mathscr{L}}$ selects elements of A^{κ} and defined new basic operations.

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   such that $\mathcal{G}$ is naturally isomorphic to $\theta_{\mathscr{L}} \circ [\kappa]$.
2. Every functor of the form $\theta_{\mathscr{L}} \circ [\kappa] \colon \mathsf{Y} \to \mathsf{X}$ is a right adjoint.

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Translations Between Languages

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Consider a cardinal $\kappa > 0$. A κ-translation of $\mathscr{L}_\mathsf{X}$ into $\mathscr{L}_\mathsf{Y}$ is a map $\boldsymbol{\tau}\colon \mathscr{L}_\mathsf{X} \to \mathscr{L}_\mathsf{Y}^\kappa$ that preserves arities.

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$$\Phi \longmapsto \{\boldsymbol{\tau}(\epsilon)(i) \approx \boldsymbol{\tau}(\delta)(i) : i < \kappa \text{ and } \epsilon \approx \delta \in \Phi\}.$$

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A translation of $\vDash_{\mathsf{X}}$ into $\vDash_{\mathsf{Y}}$ is a pair $\langle \boldsymbol{\tau}, \Theta \rangle$ where $\boldsymbol{\tau}$ is a κ-translation of $\mathscr{L}_{\mathsf{X}}$ into $\mathscr{L}_{\mathsf{Y}}$ and a set of equations Θ of Y in κ-many variables that satisfies the following conditions:

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1. For every set of equations $\Phi \cup \{\epsilon \approx \delta\}$:

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- Moreover $\langle \boldsymbol{\tau}, \Theta \rangle$ is a translation of $\vDash_{\mathsf{HA}}$ into $\vDash_{\mathsf{IA}}$.

From Translations to Right Adjoints

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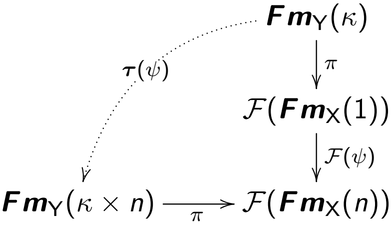

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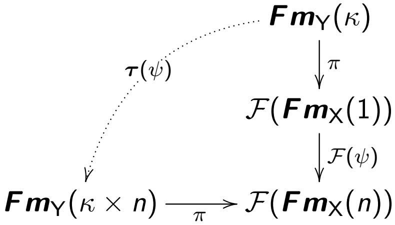

Lemma

The pair $\langle \boldsymbol{\tau}, \Theta \rangle$ is a translation of $\vDash_{\mathsf{X}}$ into $\vDash_{\mathsf{Y}}$.

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- Computational aspects: the problem of determining whether two finite algebras are related by an adjunction is decidable.

Finally...

...thank you for coming!