Order-Based and Continuous Modal Logics

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Modalities meet Degrees

"Eat before shopping. If you go to the store hungry, you are likely to make unnecessary purchases."

American Heart Association Cookbook

Many-Valued Modal Logics

Many-valued modal logics with values in $\mathbb{R}$ fall loosely into two families:

- **Order-based modal logics** (e.g., Gödel modal logics)
- **Continuous modal logics** (e.g., Łukasiewicz modal logics)

Key problems include finding axiomatizations and algebraic semantics, and establishing decidability and complexity results.

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Order-Based Algebras

Let us say that an algebra $\mathbf{A} = \langle A, \wedge, \vee, 0, 1, \ldots \rangle$ is **order-based** if

(a) $\langle A, \wedge, \vee, 0, 1 \rangle$ is a complete sublattice of $\langle [0,1], \min, \max, 0, 1 \rangle$.

(b) Each operation of $\mathbf{A}$ is definable by a quantifier-free first-order formula in a language with operations $\wedge$, $\vee$, and constants of $\mathbf{A}$.

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A Definable Operation

The **Gödel implication**

$$a \to b \;=\; \begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise} \end{cases}$$

can always be defined by the quantifier-free first-order formula

$$F^{\to}(x, y, z) \;=\; ((x \leq y) \Rightarrow (z \approx 1)) \;\&\; ((y < x) \Rightarrow (z \approx y)).$$

That is, for all $a, b, c \in A$,

$$\mathbf{A} \models F^{\to}(a, b, c) \quad \Leftrightarrow \quad a \to b = c.$$

Note also that we can also define $\neg a := a \to 0$.

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Frames and Formulas

An **A-frame** $\mathcal{F} = \langle W, R \rangle$ consists of

- a non-empty set of **states** W
- an **A**-valued **accessibility relation** $R \colon W \times W \to A$.

$\mathcal{F}$ is called **crisp** if also $Rxy \in \{0, 1\}$ for all $x, y \in W$.

We extend the language of **A** with unary (modal) connectives $\Box, \Diamond$ and define the set of formulas Fm inductively as usual.

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Models

An **A-model** $\mathcal{M} = \langle W, R, V \rangle$ adds a map $V \colon \mathrm{Fm} \times W \to A$ satisfying

$$V(\star(\varphi_1, \ldots, \varphi_n), x) \; = \; \star^{\mathbf{A}}(V(\varphi_1, x), \ldots, V(\varphi_n, x))$$

for each operation symbol $\star$ of **A**, and

$$V(\Box\varphi, x) \; = \; \bigwedge\{Rxy \to V(\varphi, y) : y \in W\}$$

$$V(\Diamond\varphi, x) \; = \; \bigvee\{Rxy \wedge V(\varphi, y) : y \in W\}.$$

$\mathcal{M}$ is called **crisp** if $\langle W, R \rangle$ is crisp, in which case,

$$V(\Box\varphi, x) \; = \; \bigwedge\{V(\varphi, y) : Rxy\}$$

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Validity

A formula φ is called

- **valid** in an **A**-model $\langle W, R, V\rangle$ if $V(\varphi, x) = 1$ for all $x \in W$
- $\mathrm{K}(\mathbf{A})$-**valid** if it is valid in all **A**-models
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Standard Gödel Modal Logics

Consider the standard algebra for **Gödel logic**

$$\mathbf{G} = \langle [0,1], \wedge, \vee, \rightarrow, 0, 1 \rangle.$$

An axiomatization for $\mathrm{K}(\mathbf{G})$ is obtained by adding the prelinearity axiom schema $(\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$ to the intuitionistic modal logic IK.

X. Caicedo and R. Rodríguez.
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Other Gödel Modal Logics

More generally, we may consider (expansions of) **Gödel modal logics** $\mathrm{K}(\mathbf{A})$ and $\mathrm{K}(\mathbf{A})^{\mathrm{C}}$ where $\mathbf{A}$ is any complete subalgebra of $\mathbf{G}$; e.g.,

$$A = \{0\} \cup \{\tfrac{1}{n+1} \mid n \in \mathbb{N}\} \quad \text{or} \quad A = \{1 - \tfrac{1}{n+1} \mid n \in \mathbb{N}\} \cup \{1\}.$$

Indeed, there are countably infinitely many different infinite-valued Gödel modal logics (considered as sets of valid formulas).

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Failure of the Finite Model Property

The following formula is valid in all **finite** $\mathrm{K}(\mathbf{G})$-models

$$\Box\neg\neg p \to \neg\neg\Box p$$

but not in the **infinite** $\mathrm{K}(\mathbf{G})$-model $\langle \mathbb{N}, \mathbb{N}^2, V \rangle$ where $V(p,x) = \frac{1}{x+1}$.

$$\Big(V(\Box\neg\neg p \to \neg\neg\Box p, 0) \;=\; \big(\bigwedge_{x\in\mathbb{N}} V(\neg\neg p, x)\big) \to \big(\neg\neg \bigwedge_{x\in\mathbb{N}} V(p,x)\big)$$

$$= \big(\bigwedge_{x\in\mathbb{N}} 1\big) \to \big(\neg\neg \bigwedge_{x\in\mathbb{N}} \tfrac{1}{x+1}\big)$$

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Towards Decidability

We prove decidability (indeed PSPACE-completeness) for order-based modal logics satisfying a certain topological property by providing new semantics that admit the finite model property.

X. Caicedo, G. Metcalfe, R. Rodríguez, and J. Rogger.
Decidability of Order-Based Modal Logics.
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The idea is to restrict the values at each state that can be taken by box and diamond formulas; $\Box\varphi$ and $\Diamond\varphi$ can then be "witnessed" at states where the value of φ is "sufficiently close" to the value of $\Box\varphi$ or $\Diamond\varphi$.

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A New Semantics

We augment **G**-frames with a map T from states to **finite** subsets of $[0, 1]$ containing 0 and 1, and **G**-models are defined as before except that

$$V(\Box\varphi, x) = \max\{r \in T(x) : r \leq \bigwedge_{y \in W} (Rxy \to V(\varphi, y))\}$$

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A Finite Counter Model

We find a **finite** counter-model for $\Box\neg\neg p \to \neg\neg\Box p$:

$$\langle \{a\}, \{(a,a)\}, T, V\rangle \qquad \text{where} \quad V(p,a) = \tfrac{1}{2} \quad \text{and} \quad T(a) = \{0,1\}.$$

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More Generally...

We consider an order-based algebra **A** that is "locally homogeneous"; roughly, for any right (or left) accumulation point a of **A**, there is an interval $[a, c)$ (or $(c, a]$) that can be squeezed without changing the order.

We augment an **A**-frame $\langle W, R \rangle$ with maps

$$T_\Box \colon W \to \mathcal{P}(A) \quad \text{and} \quad T_\Diamond \colon W \to \mathcal{P}(A)$$

such that for each $x \in W$,

- the constants of **A** are contained in both $T_\Box(x)$ and $T_\Diamond(x)$
- $T_\Box(x) = A \setminus \bigcup_{i \in I}(a_i, c_i)$ for some finite I, where each $c_i \in A$ witnesses homogeneity at a right accumulation point a_i of **A**
- $T_\Diamond(x) = A \setminus \bigcup_{j \in J}(d_j, b_j)$ for some finite J, where each $d_j \in A$ witnesses homogeneity at a left accumulation point b_j of **A**.

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More Generally...

We consider an order-based algebra **A** that is "locally homogeneous"; roughly, for any right (or left) accumulation point a of **A**, there is an interval $[a, c)$ (or $(c, a]$) that can be squeezed without changing the order.

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Decidability and Complexity

For any locally homogeneous order-based algebra **A**:

- $\mathrm{K}(\mathbf{A})$ and $\mathrm{K}(\mathbf{A})^{\mathrm{C}}$ are sound and complete with respect to the new semantics.
- The new semantics has the finite model property.
- If there is an oracle for checking consistency with finite models, then validity in $\mathrm{K}(\mathbf{A})$ and $\mathrm{K}(\mathbf{A})^{\mathrm{C}}$ are both decidable.

In particular, validity in $\mathrm{K}(\mathbf{G})$ and $\mathrm{K}(\mathbf{G})^{\mathrm{C}}$ are both PSPACE-complete.

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Beyond the Basic Modal Logics

- We have also obtained decidability (indeed, co-NP completeness) for order-based modal logics $\mathrm{S5}(\mathbf{A})^{\mathrm{C}}$ based on crisp $\mathrm{K}(\mathbf{A})$-models where R is an equivalence relation.
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Continuous Modal Logics

Łukasiewicz modal logics are defined with connectives on $[0, 1]$

$$\begin{array}{rclcrcl} x \to y & = & \min(1, 1 - x + y) & \qquad & \neg x & = & 1 - x \\ x \oplus y & = & \min(1, x + y) & \qquad & x \odot y & = & \max(0, x + y - 1). \end{array}$$

Łukasiewicz (multi-)modal logics can also be viewed as fragments of **continuous logic** and studied as **fuzzy description logics**.

Using the fact that Łukasiewicz modal logics enjoy the finite model property, it can be shown that validity in these logics is in $2\mathit{EXPTIME}$.

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Hansoul and Teheux (2013) axiomatize Łukasiewicz modal logic over crisp Kripke frames by adding to an axiomatization of Łukasiewicz logic

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We have axiomatized a modal logic over $\mathbb{R}$ with abelian group operations (extending the multiplicative fragment of Abelian logic), whose validity problem is in EXPTIME.

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Challenges

- Find an axiomatization of crisp Gödel modal logic.
- Develop a robust algebraic theory for order-based modal logics.
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