A representation for the n-generated free algebra in the subvariety of BL-algebras generated by $[0,1]_{\mathbf{MV}} \oplus [0,1]_{\mathbf{G}}$

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Examples of standard algebras

Standard MV-algebra $[0,1]_{\mathbf{MV}}$:

$$\left\langle [0,1], \begin{cases} 0 & \text{if } x+y \leq 1 \\ x+y-1 & \text{oherwise} \end{cases}, \begin{cases} 1 & \text{if } x \leq y \\ 1-x+y & \text{otherwise} \end{cases}, 0 \right\rangle$$

Standard Gödel-algebra $[0,1]_{\mathbf{Gödel}}$:

$$\left\langle [0,1], \begin{cases} x & \text{if } x \leq y \\ y & \text{oherwise} \end{cases}, \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}, 0 \right\rangle$$

Examples of free algebras: the case of MV-algebras

Chang's Algebraic Completeness Theorem

The standard MV-algebra $\langle [0,1], max(0, x+y-1), min(1, 1-x+y), 0 \rangle$ *is generic for the variety of MV-algebras (BL algebras with* $\neg\neg x = x$*).*

Consider the MV-algebra $\mathcal{M}_n$ of all functions $f : [0,1]^n \to [0,1]$ endowed with the pointwise standard MV-operations:
$(f \cdot g)(x) = max(0, f(x) + g(x) - 1)$,
$(f \to g)(x) = min(1, 1 - f(x) + g(x))$, $\bot(x) = 0$.

McNaughton's Representation Theorem

The free n-generated MV-algebra is the subalgebra of $\mathcal{M}_n$ *of all continuous piecewise linear functions* $f : [0,1]^n \to [0,1]$ *where each one of the finitely many linear pieces has integer coefficients.*

Examples of free algebras: the case of Gödel hoops

Gödel hoops are the $\bot$-free subreducts of Gödel algebras.
Gödel hoop form a variety $\mathbf{G}$.
We will call $[0,1]_{\mathbf{G}}$ to the standard Gödel hoop.

Definition

Let $\mathcal{R}$ be the set which contains all the subsets of $[0,1]^n$ given by:

$$R \in \mathcal{R} \text{ iff } R = \{(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) : x_{\sigma(1)} \Box \ldots \Box x_{\sigma(n)}\}$$

for $\Box \in \{=, <\}$ and σ a permutation of $\{1, \ldots, n\}$.

Free n-generated Gödel hoops

Theorem: The case for Gödel algebras

The algebra of functions $f : [0,1]^n \to [0,1]$ *such that for every* $R \in \mathcal{R}$

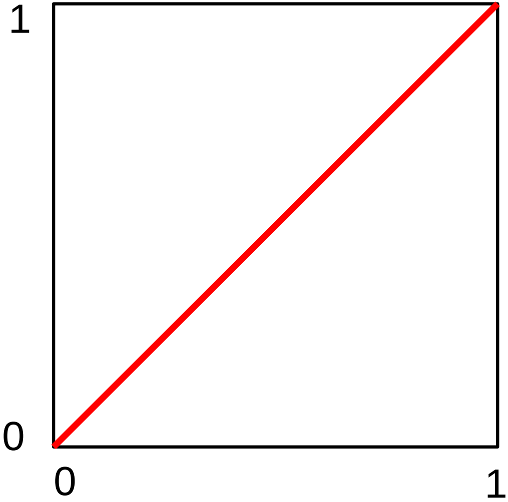
$$f|_R = 1, \text{ or } f|_R = 0, \text{ or}$$
$$f|_R = x_i \text{ with } i \in \{1, \ldots, n\}$$

equipped with the pointwise operations $\cdot$ *and* $\to$ *is the free Gödel algebra over n-generators.* [1]

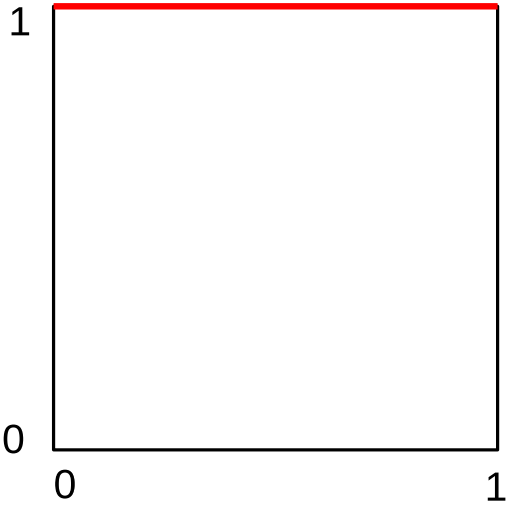
For the case of the free Gödel hoops algebra $Free_{\mathcal{G}}(n)$ it also holds: $f > 0$ and if $f|_R = x_i$ where R is the region defined by $x_{\sigma(1)} \Box \ldots < x_{\sigma(i)} \Box \ldots \Box x_{\sigma(n)}$ then $f|_S = x_i$ for every $S \in \mathcal{R}$ where S is a region where the last $n - i$ variables are ordered as in R.

[1] B. Gerla, Many valued Logics of Continuous t-norms and their Functional Representation, PhD thesis, Università di Milano, 2000/2001

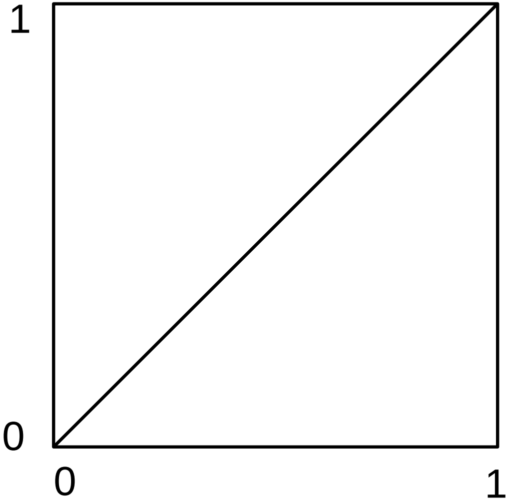
The case of one variable



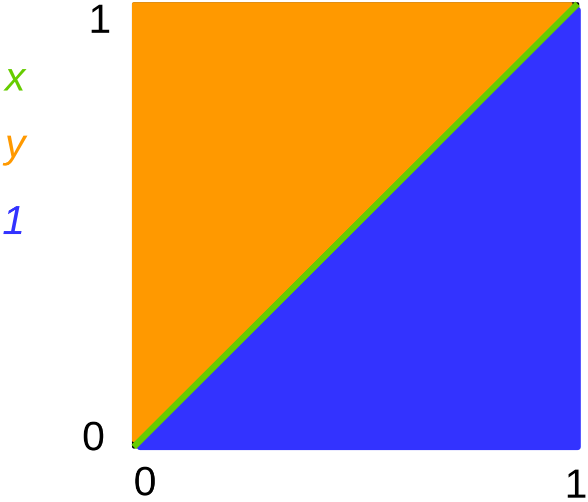
The case of one variable



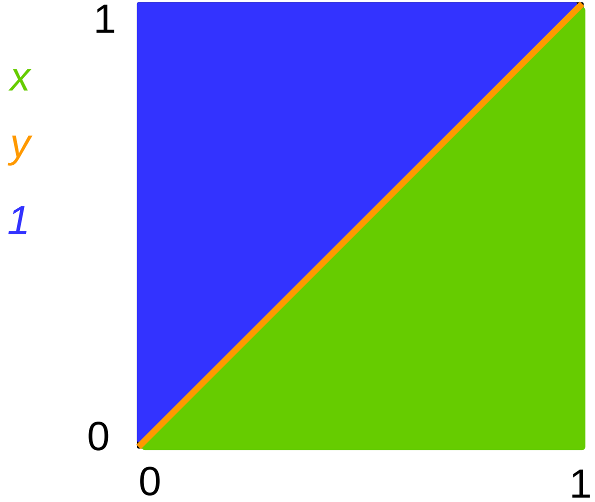
The case of two variables



The case of two variables



The case of two variables



Ordinal sum

Let $\mathbf{R} = (R, *_{\mathbf{R}}, \rightarrow_{\mathbf{R}}, \top)$ and $\mathbf{S} = (R, *_{\mathbf{S}}, \rightarrow_{\mathbf{S}}, \top)$ be two hoops such that $R \cap S = \{\top\}$. We define the ordinal sum $R \oplus S$ of these two hoops as the hoop given by $(R \cup S, *, \rightarrow, \top)$ where the operations $(*, \rightarrow)$ are defined as follows:

$$x * y \begin{cases} x *_{\mathbf{R}} y & \text{if } x, y \in R, \\ x *_{\mathbf{S}} y & \text{if } x, y \in S, \\ x & \text{if } x \in R \setminus \{\top\} \text{ and } y \in S, \\ y & \text{if } y \in R \setminus \{\top\} \text{ and } x \in S. \end{cases}$$

$$x \rightarrow y \begin{cases} \top & \text{if } x \in R \setminus \{\top\} \text{ and } y \in S, \\ x \rightarrow_{\mathbf{R}} y & \text{if } x, y \in R, \\ x \rightarrow_{\mathbf{S}} y & \text{if } x, y \in S, \\ y & \text{if } y \in R \setminus \{\top\} \text{ and } x \in S. \end{cases}$$

- $Free_{\mathcal{BL}}(n)$ is generated by the algebra $(n+1)[0,1]_{\mathbf{MV}}$. This fact allows us to characterize the free n-generated BL-algebra $Free_{\mathcal{BL}}(n)$ as the algebra of functions $f : (n+1)[0,1]^n_{\mathbf{MV}} \to (n+1)[0,1]_{\mathbf{MV}}$ generated by the projections.
- S. Bova and S. Aguzzoli gave a representation of the free-n-generated BL-algebra. [2], [3]

In this work we will concentrate in the subvariety $\mathcal{V} \subseteq \mathcal{BL}$ generated by the ordinal sum of the algebra $[0,1]_{\mathbf{MV}}$ and the Gödel hoop $[0,1]_{\mathbf{G}}$, that is, generated by $\mathbf{A} = [0,1]_{\mathbf{MV}} \oplus [0,1]_{\mathbf{G}}$.

[2] S. Bova, PhD thesis, BL-functions and Free BL-algebra,2008

[3] S. Aguzzoli and S. Bova, The free n-generated BL-algebra, Ann. Pure Appl. Logic, Vol. 161, 9, p.1144–1170, 2010

Some remarks...

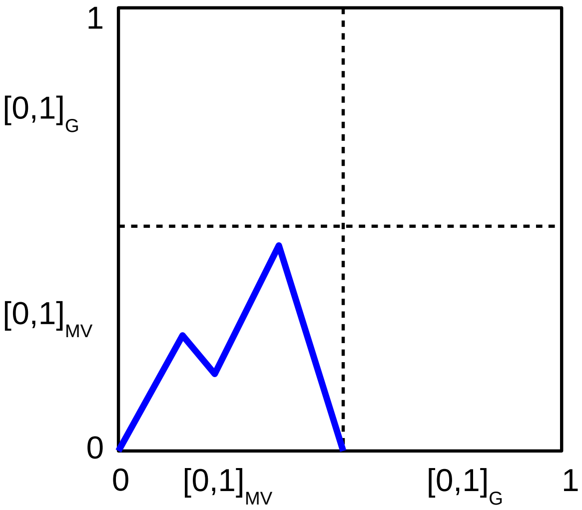
- $[0,1]_{\mathbf{G}}$ is decomposable as an infinite ordinal sum of two-elements Boolean algebra, the idea is to treat it as a whole block (dense elements).
- The elements in $[0,1]_{\mathbf{MV}}$ are usually called regular elements of $\mathbf{A}$.
- **Advantage**: The number n of generators of the free algebra does not increase the generating chain.
- That gives an idea of the role of the regular elements and the role of the dense elements.
- To give a functional representation for the free algebra $Free_{\mathcal{V}}(n)$ we decompose the domain $[0,1]_{\mathbf{MV}} \oplus [0,1]_{\mathbf{G}}$ in a finite number of pieces. In each piece a function $F \in Free_{\mathcal{V}}(n)$ coincides either with McNaughton functions or functions on the free algebra in the variety of Gödel hoops.

$Free_{\mathcal{V}}(1)$

Proposition

Let $\alpha(x)$ be a BL-term in one variable that we evaluate in $\mathcal{V}$. Then:

- *If $\alpha_{\mathcal{V}}(1) = 1$ then $\alpha_{\mathcal{V}}(x)$ is a function of $Free_{\mathcal{G}}(1)$ for each $x \in [0, 1]_{\mathbf{G}}$.*
- *If $\alpha_{\mathcal{V}}(1) = 0$ then $\alpha_{\mathcal{V}}(x) = 0$ for each $x \in [0, 1]_{\mathbf{G}}$.*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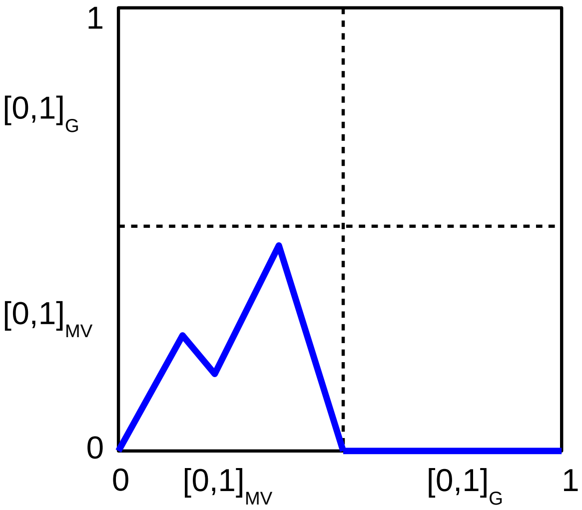
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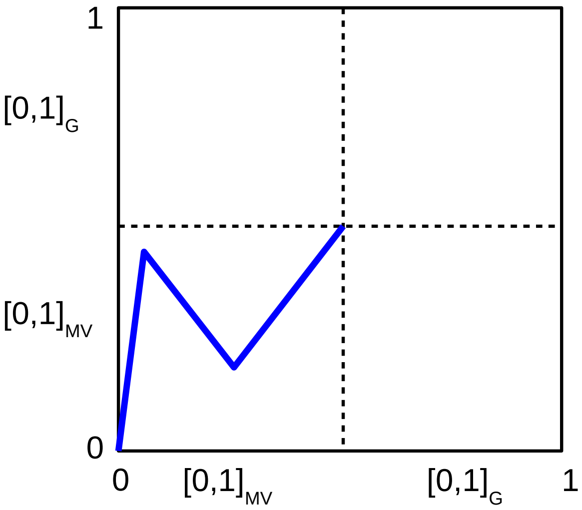
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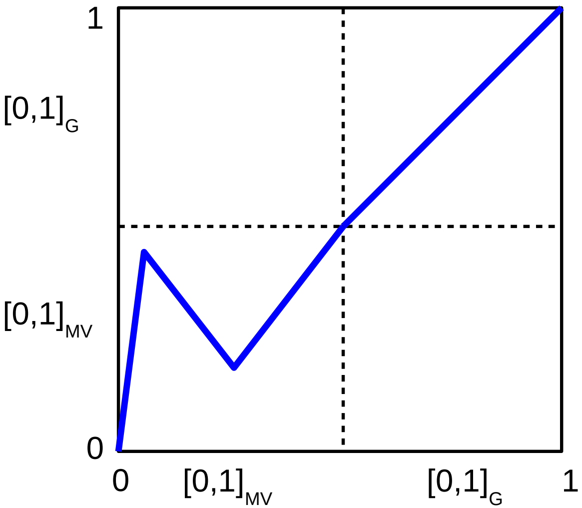
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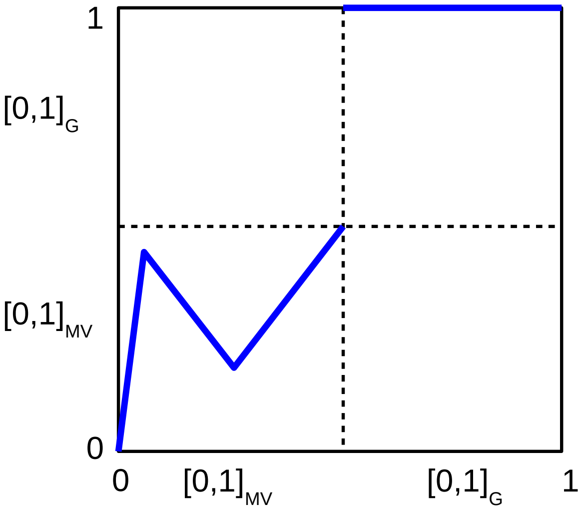
1
[0,1]G
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[0,1]G
1

Proposition

Let $g \in Free_{\mathcal{MV}}(1)$ *and* $h \in Free_{\mathcal{G}}(1)$ *such that* $g(1) = h(1) = 1$. *Then the function*

$$f(x) = \begin{cases} g(x) & \text{if} \quad x \in [0,1]_{\mathbf{MV}} \\ h(x) & \text{if} \quad x \in [0,1]_{\mathbf{G}} \end{cases} \tag{1}$$

is in $Free_{\mathcal{V}}(1)$.

Proposition

Let $g \in Free_{\mathcal{MV}}(1)$ *such that* $g(1) = 0$. *Then the function*

$$f(x) = \begin{cases} g(x) & \text{if} \quad x \in [0,1]_{\mathbf{MV}} \\ 0 & \text{if} \quad x \in [0,1]_{\mathbf{G}} \end{cases} \tag{2}$$

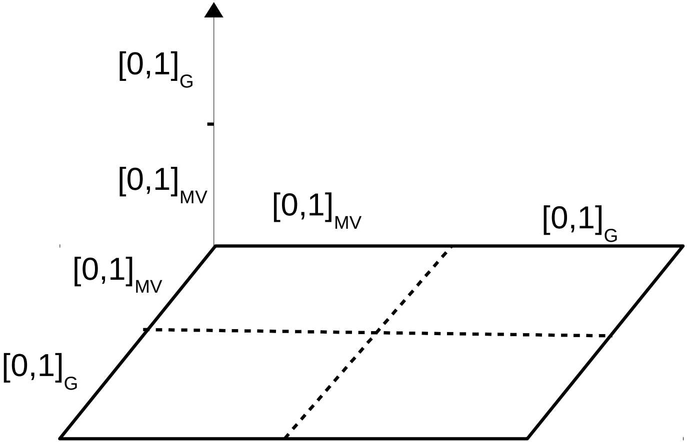
is in $Free_{\mathcal{V}}(1)$.

Proposition

Let $f \in Free_{\mathcal{V}}(1)$.

- *If $f(1) = 1$ then there are functions $g \in Free_{\mathcal{MV}}(1)$ and $h \in Free_{\mathcal{G}}(1)$ such that satisfy (1).*
- *If $f(1) = 0$ then there is a function $g \in Free_{\mathcal{MV}}(1)$ such that satisfies (2).*

Problem in two variables



As before, if $\alpha(x, y)$ is a BL-term and we evaluate it in $\mathcal{V}$ we have:

- If $\alpha_{\mathcal{V}}(1, 1) = 1$ then there is a function $g \in Free_{Free_{\mathcal{G}}}(2)$ such that $\alpha_{\mathcal{V}}(x, y) = g(x, y)$ for every $(x, y) \in [0, 1]^2_G$.
- If $\alpha_{\mathcal{V}}(1, 1) = 0$ then $\alpha_{\mathcal{V}}(x, y) = 0$ for every $(x, y) \in [0, 1]^2_G$.

Proposition

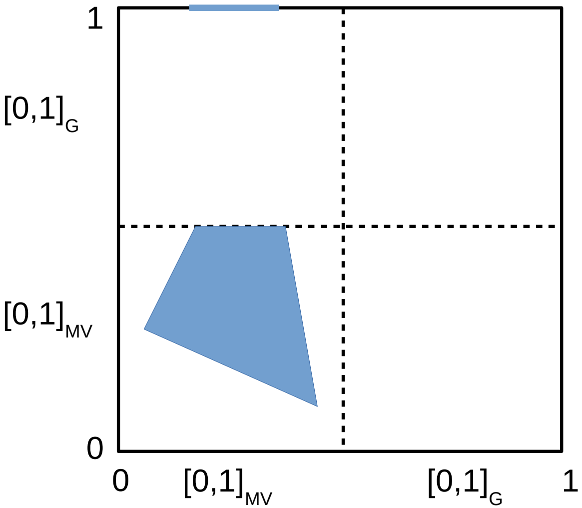
Let $\alpha(x, y)$ *and* $a \in [0,1]_{MV} \setminus \{1\}$. *Then, if we evaluate* α *on* $\mathcal{V}$, *it holds:*

- *If* $\alpha_{\mathcal{V}}(a, 1) = c \in [0,1]_{MV} \setminus \{1\}$ *then* $\alpha_{\mathcal{V}}(a, b) = c$ *for every* $b \in [0,1]_G$,
- *If* $\alpha_{\mathcal{V}}(a, 1) = 1$ *then there is a function* $g \in Free_{\mathcal{V}}(1)$ *such that* $\alpha_{\mathcal{V}}(a, b) = g(b)$ *for every* $b \in [0,1]_G$.

Definition

Let $f \in Free_{\mathcal{MV}}(2)$. If $A = \{x \in [0,1]_{MV} : f(x,1) = 1\}$ and $B = [0,1]_{MV} \setminus A$, we will say that $g : [0,1]_{MV} \times (0,1]_G \to \mathcal{V}$ is an f-y-G-McNaughton function if:

1. For each $x_0 \in B$, $g(x_0, y) = f(x_0, 1)$, for every $y \in (0,1]_G$.
2. There is a regular triangulation Δ of A which determines the simplexes $\sigma_1, \ldots, \sigma_n$ and functions $g_1, \ldots, g_n \in Free_{\mathcal{G}}(1)$ such that $g(x,y) = g_i(y)$, for every x in the interior of σ_i.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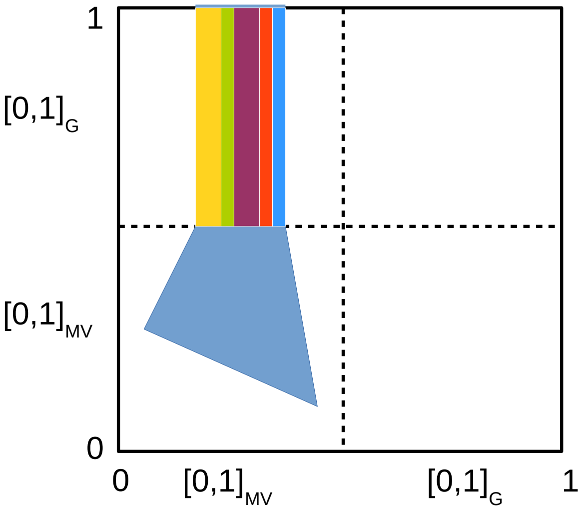
[0,1]G
1


1
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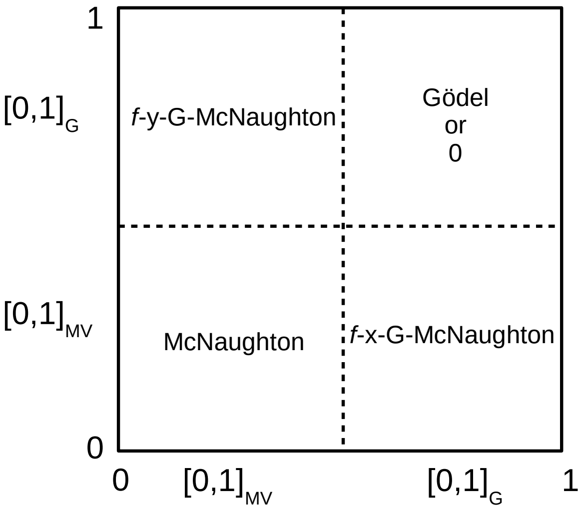

1
[0,1]_G
f-y-G-McNaughton
Gödel
or
0
[0,1]_MV
McNaughton
f-x-G-McNaughton
0
0
[0,1]_MV
[0,1]_G
1

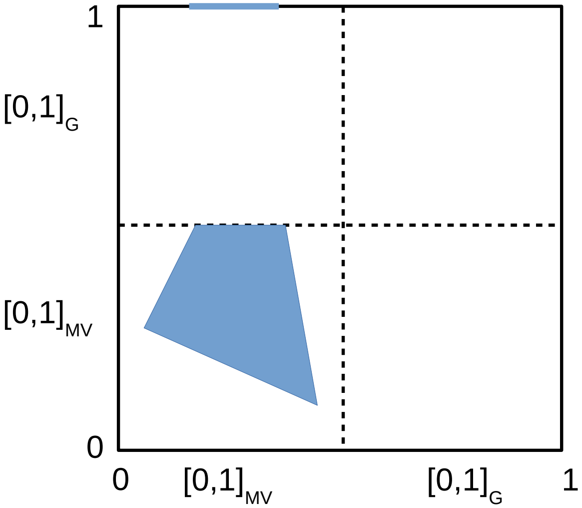
Open intervals

Lemma

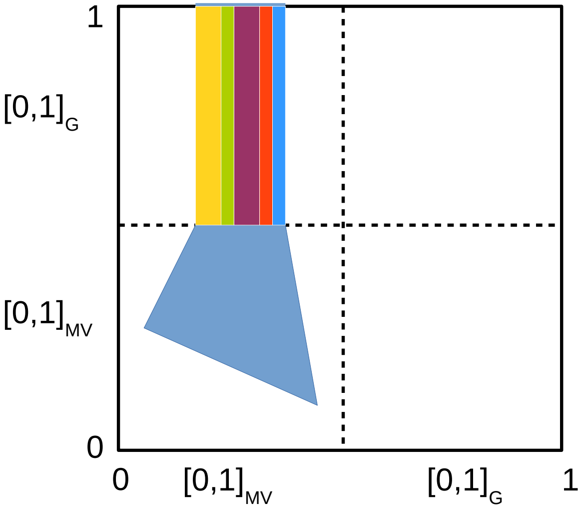
If $g \in Free_{\mathcal{G}}(1)$ and $S \subseteq [0,1]_{MV}$ is an open interval with rational borders, then there is a term γ_S in two variables such that the interpretation of the term on $\mathcal{V}$ satisfies:

$$\gamma_S(x,y) = \begin{cases} g(y) & \text{if } (x,y) \in S \times [0,1]_G \\ 1 & \text{otherwise.} \end{cases} \tag{3}$$

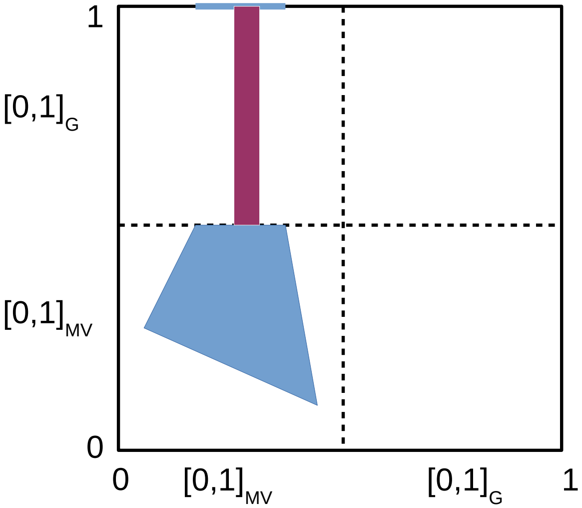
Geometrical idea of the proof



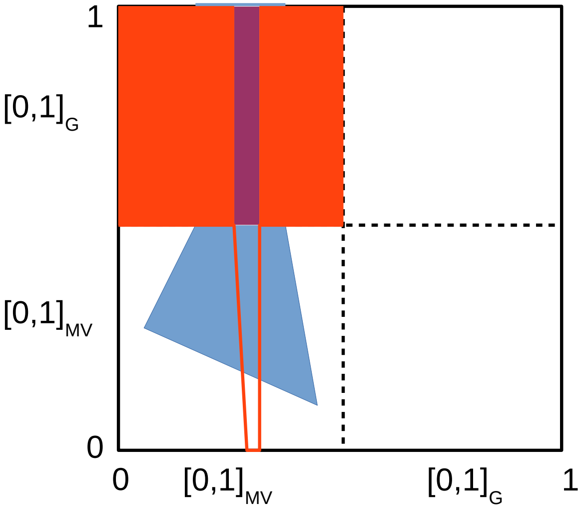
Geometrical idea of the proof



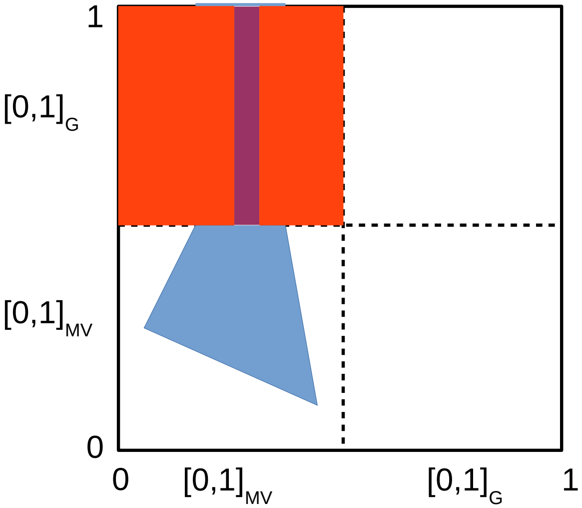
Geometrical idea of the proof



Geometrical idea of the proof



Geometrical idea of the proof



Corollary

Let $f \in Free_{\mathcal{MV}}(2)$ *and* h_x *an* f*-*x*-G-McNaughton function. If* Δ *is the triangulation of* $[0,1]_{MV} \times \{1\} \cap f^{-1}(\{1\})$ *given in the definition of* h_x *and for every simplex* $\sigma_i \in \Delta$ *we denote* σ_i^0 *to the relative interior of the simplex, and* $g_i \in Free_{\mathcal{G}}(1)$ *also are de functions given in the definition of* h_x*, then there is a function* $F_x \in Free_{\mathcal{V}}(2)$ *which satisfies:*

$$F_x(x,y) = \begin{cases} g_i(y) & \text{if } (x,y) \in \sigma_i^0 \times [0,1]_G \\ 1 & \text{otherwise} \end{cases}$$

Lemma

Given a function $g \in Free_{\mathcal{G}}(2)$ there is a function $f_g \in Free_{\mathcal{V}}(2)$ which satisfies:

$$f_g(x,y) = \begin{cases} g(x,y) & \text{if } (x,y) \in (0,1]_G \times (0,1]_G \\ 1 & \text{otherwise} \end{cases}$$

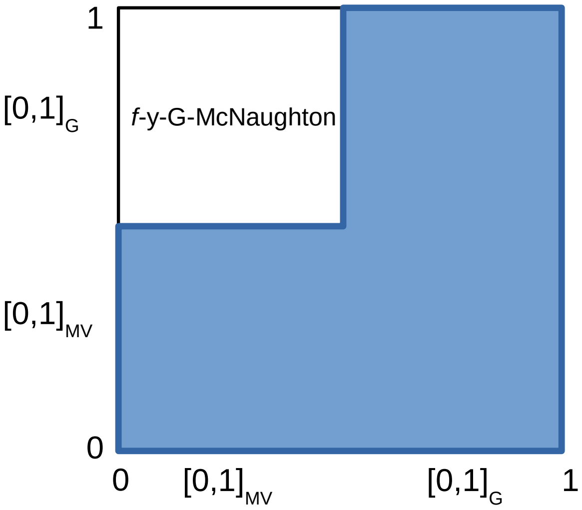
Let $F : \mathcal{V}^2 \to V$ given as before.
Consider the terms:

1. $\gamma_1 = \alpha$
2. γ_2 is a term whose interpretation on A is the function F_x correspondent to the f-x-G-McNaughton function h_x.
3. γ_3 is a term whose interpretation on A is the function F_y correspondent to the f-y-G-McNaughton function h_y.
4. γ_4 is a term whose interpretation on A is the function f_g correspondent to the function $g \in Free_{\mathcal{G}}(2)$.

We define the two-variables term β given by

$$\beta = \bigwedge_{i=1}^{4} \gamma_i.$$

Then the interpretation of β in the algebra $[0,1]_{\mathbf{MV}} \oplus [0,1]_{\mathbf{G}}$ coincides with the function F.



1
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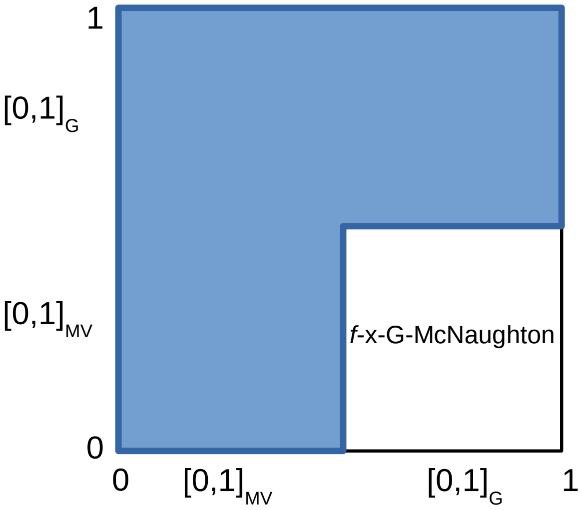
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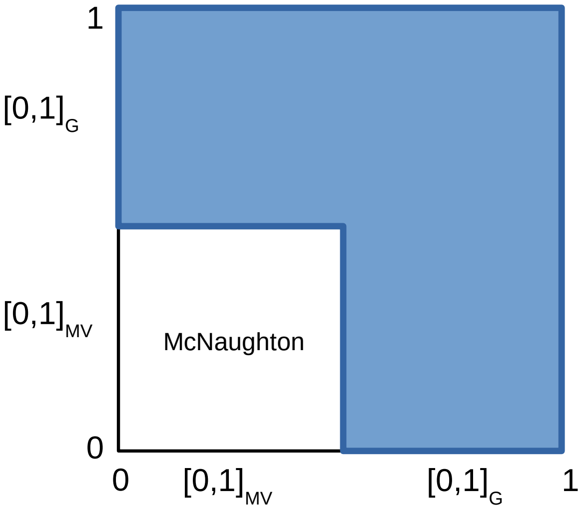
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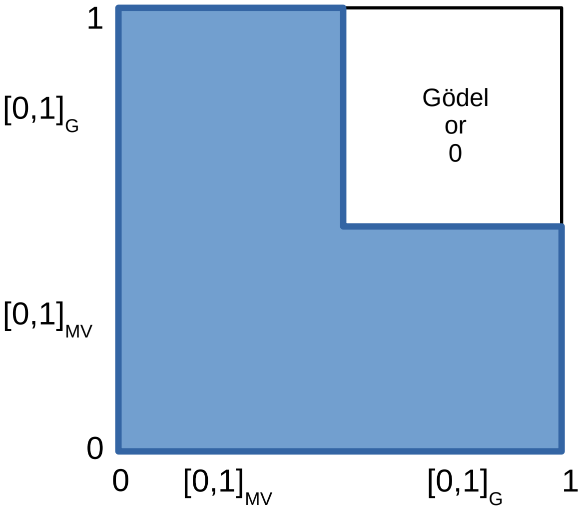
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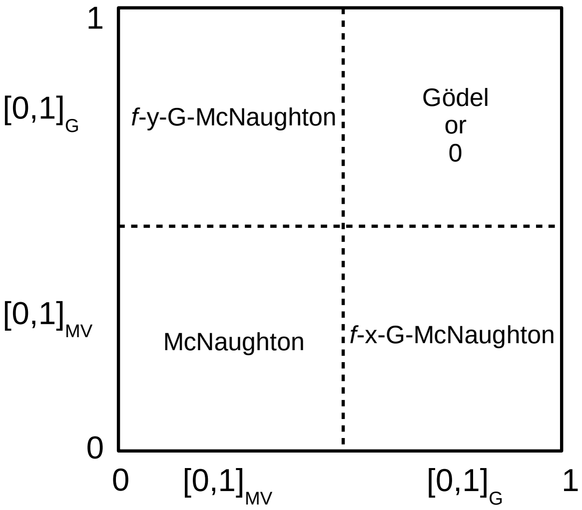
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[0,1]$_{MV}$
[0,1]$_G$
1

$Free_{\mathcal{V}}(n)$

Let $F \in Free_{\mathcal{V}}(n)$. Then:

- For every $\bar{x} \in ([0,1]_{\mathbf{MV}})^n$,

$$F(\bar{x}) = f(\bar{x})$$

  where f is a function of $Free_{\mathcal{MV}}(n)$.

For the rest of the domain, the functions depend on this function $f : ([0,1]_{\mathbf{MV}})^n \to [0,1]_{\mathbf{MV}}$:

- On $([0,1]_{\mathbf{G}})^n$:
  1. If $f(\bar{1}) = 0$, then
  $$F(\bar{x}) = 0$$
  for every $\bar{x} \in ([0,1]_{\mathbf{G}})^n$.
  2. If $f(\bar{1}) = 1$, then
  $$F(\bar{x}) = g(\bar{x})$$
  for a function $g \in Free_{\mathcal{G}}(n)$, for every $\bar{x} \in ([0,1]_{\mathbf{G}})^n$.

Let $B = \{x_{\sigma(1)}, \dots x_{\sigma(m)}\} \subsetneq \{x_1, \dots, x_n\}$ and R_B be the subset of $([0,1]_{MV} \oplus [0,1]_G)^n$ where $x_i \in B$ if and only if $x_i \in [0,1]_G$. For every $\bar{x} \in R_B$ we also define $\tilde{x}$ as:

$$\tilde{x}_i = \begin{cases} x_i & \text{if} \quad x_i \notin B \\ \\ 1 & \text{if} \quad x_i \in B \end{cases}$$

- 1. If $f(\tilde{x}) < 1$ then $F(\bar{x}) = f(\tilde{x})$.
  2. If $f(\tilde{x}) = 1$, then there is a regular triangulation Δ of $f^{-1}(1) \wedge R_B$ which determines the simplices $S_1, \dots, S_k$ and k Gödel functions $h_1, \dots, h_n$ in $n - m$ variables $x_{\sigma(m+1)}, \dots, x_{\sigma(n)}$ such that $F(\bar{x}) = h_i(x_{\sigma(m+1)}, \dots, x_{\sigma(n)})$ for each point $(x_{\sigma(1)}, \dots x_{\sigma(m)})$ in the interior of S_i.

Thank you!