A NEW LOOK AT THE EFFECTS OF A HILBERT SPACE

R. Giuntini, A. Ledda, F. Paoli

Università di Cagliari, Italia

EVENT-STATE SYSTEM

$\downarrow \quad \downarrow$

$$\langle \Pi(\mathcal{H}) \, , \, \mathcal{S}(\mathcal{H}) \rangle$$

- $\Pi(\mathcal{H})$ is the set of all projections of the Hilbert space $\mathcal{H}$.
- $\mathcal{S}(\mathcal{H})$ is the set of all density operators of $\mathcal{H}$.

The set of all projections is in 1-1 correspondence with the set of all closed subspaces of $\mathcal{H}$.

OBSERVABLES

are represented as

PROJECTION-VALUED MEASURES

- $\forall P, Q \in \Pi(\mathcal{H})$:
  - $P \wedge Q$ is the projection onto the closed subspace associated to the intersection of the closed subspaces that are associated to the projections P and Q;
  - $P \vee Q$ is the projection onto the smallest closed subspace associated to the union of the closed subspaces that are associated to the projections P and Q.
  - $P' = \mathbb{I} - P$, where $\mathbb{I}$ is the identity operator. Equivalently, if X is the closed subspace associated to P, then P' is the projection that is associated to the closed subspace $X' := \{\psi \in \mathcal{H} \mid \forall \varphi \in X : \psi \perp \varphi\}$.

$$\langle \Pi(\mathcal{H}), \wedge, \vee, {}', \mathbb{O}, \mathbb{I} \rangle$$

is an orthomodular lattice, where the induced partial order $\preceq$ turns out to be:

$$P \preceq Q \text{ iff } \forall \psi \in \mathcal{H} : \langle \psi | P\psi \rangle \leq \langle \psi | Q\psi \rangle.$$

Some features of projections

- $\forall P \in \Pi(\mathcal{H})$: $\quad Eigenvalues(P) \subseteq \{0, 1\}$;
- $\forall P \in \Pi(\mathcal{H})$: $\quad P \vee P' = \mathbb{I}$ (excluded middle law, sharpness).
- $\forall P \in \Pi(\mathcal{H})$: $\quad P \wedge P' = \mathbb{O}$ (noncontradiction law, sharpness).
- $P' = P_{ker(P)}$, where $P_{ker(P)}$ is the projection that is associated to the closed subspace $ker(P) := \{\psi \in \mathcal{H} \mid P\psi = \underline{0}\}$;
- $d(\mathcal{H}) < \infty$ iff $\Pi(\mathcal{H})$ is modular, i.e.,

$$P \preceq Q \text{ iff } \forall R \in \Pi(\mathcal{H}): P \vee (Q \wedge R) = Q \wedge (P \vee R).$$

The notion of **quantum event** is liberalized.

The set $\Pi(\mathcal{H})$ is replaced by the set of all effects of $\mathcal{H}$ (denoted by $\mathcal{E}(\mathcal{H})$), where an effect of $\mathcal{H}$ is a bounded linear operator E that satisfies the following condition:

$$\forall \rho \in \mathcal{S}(\mathcal{H}) : \ \mathtt{tr}(\rho E) \in [0,1].$$

In general: $E^2 \neq E$. Thus, $\Pi(\mathcal{H}) \subset \mathcal{E}(\mathcal{H})$.

OBSERVABLES

are represented as

POSITIVE-OPERATOR VALUED MEASURES

- $\mathcal{E}(\mathcal{H})$ can be partially ordered:

$$E \preceq F \text{ iff } \forall \rho \in \mathcal{S}(\mathcal{H}): \ \mathtt{tr}(\rho E) \leq \mathtt{tr}(\rho F).$$

$$\Downarrow$$

$$\mathcal{E}(\mathcal{H}) = \{E \mid \mathbb{O} \preceq E \preceq \mathbb{I}\}.$$

An effect is a positive linear operator bounded by $\mathbb{I}$.

- $\mathcal{E}(\mathcal{H})$ can be equipped with an involution operation $'$:

$$E' = \mathbb{I} - E.$$

Theorem

$\langle \mathcal{E}(\mathcal{H}), \preceq, ', \mathbb{O}, \mathbb{I} \rangle$ is a Kleene bounded involution poset, i.e., a bounded involution poset that satisfies the Kleene condition:

$$\forall E, F \in \mathcal{E}(\mathcal{H}) : \text{if } E \preceq E' \text{ and } F \preceq F', \text{ then } E \preceq F'.$$

However, in general:

$$E \wedge E' \neq \mathbb{O}$$

The noncontradiction law fails.

Definition

An bounded involution poset (lattice) is a structure $\mathcal{L} = \langle L\,,\,\preceq\,,\,'\,,\,\mathbf{0}\,,\mathbf{1}\rangle$ where:

- $\langle L\,,\,\preceq\,,\,\mathbf{0}\,,\,\mathbf{1}\rangle$ is a bounded poset (lattice) $(\forall a \in L:\ \mathbf{0} \preceq a$ and $a \preceq \mathbf{1})$;
- $'$ is an involution, i.e., a 1-ary operation $'\ :\ L \to\ L$ such that $\forall a, b \in L$:
  - $a'' = a$; (double negation)
  - If $a \preceq b$, then $b' \preceq a'$; (contraposition) (equivalently, $(a \vee b)' = a' \wedge b'$, provided $\mathcal{L}$ is a lattice).

- The class of all bounded involution lattices is a **variety** ($\mathbb{BIL}$).

Definition

An Kleene poset (lattice) is a bounded involution poset $\mathcal{L} = \langle L\,,\ \preceq\,,\ '\,,\ \mathbf{0}\,,\mathbf{1}\rangle$ that satisfies the Kleene condition:

$$\forall a, b \in L: \text{ if } a \preceq a' \text{ and } b \preceq b', \text{ then } a \preceq b'$$

- The class of all Kleene lattices is a **variety** ($\mathbb{KL}$), being the Kleene condition equivalent to the equation

$$a \wedge a' \preceq b \vee b'$$

$\langle \mathcal{E}(\mathcal{H}), \preceq, ', \mathbb{O}, \mathbb{I} \rangle$ is a Kleene poset

,

which is **not a lattice**.

Is it possible to equip $\mathcal{E}(\mathcal{H})$ with lattice-theoretic operations?

Definition

A (bounded) spectral family is a map $M : \mathbb{R} \to \Pi(\mathcal{H})$ that satisfies the following conditions:

- $\forall \lambda, \mu \in \mathbb{R}$: if $\lambda \leq \mu$, then $M(\lambda) \preceq M(\mu)$ (*monotonicity*);
- $\forall \lambda \in \mathbb{R}$: $M(\lambda) = \bigwedge_{\mu > \lambda} M(\mu)$ (*right-continuity*);
- $\exists \lambda, \mu \in \mathbb{R}$ such that $\forall \eta \in \mathbb{R}$:
  $$M(\eta) = \begin{cases} \mathbb{O}, & \text{if } \eta < \lambda; \\ \mathbb{I}, & \text{if } \eta \geq \mu. \end{cases}$$

Theorem

- *If* A *is a bounded self-adjoint operator of* $\mathcal{H}$*, then there exists a spectral family* M^A *such that*

$$A = \int_{-\infty}^{\infty} \lambda d M^A(\lambda), \qquad (**)$$

where the integral is meant in the sense of norm-converging Riemann-Stieltjes sums.

- *Every spectral family* $M : \mathbb{R} \to \Pi(\mathcal{H})$ *determines a* **unique** *bounded self-adjoint operator on* $\mathcal{H}$ *according to (**).*

$M^A :=$ the spectraly family **uniquely** associated to the bounded self-adjoint operator A.

In particular, if E is a projection, we have:

$$M^E(\lambda) = \begin{cases} \mathbb{O}, & \text{if } \lambda < 0; \\ \mathbb{I} - E, & \text{if } 0 \leq \lambda < 1; \\ \mathbb{I}, & \text{if } \lambda \geq 1. \end{cases}$$

Let E, F be two effects.

$$E \preceq_s F \text{ iff } \forall \lambda \in \mathbb{R}: \; M^F(\lambda) \preceq M^E(\lambda).$$

Theorem (Olson 1971, H. F. de Groote 2005)

- *$\forall E, F \in \mathcal{E}(\mathcal{H})$: $E \preceq_s F$ implies $E \preceq F$, but not the other way around;*
- *$\forall E, F \in \mathcal{E}(\mathcal{H})$: if E and F commute ($EF = FE$), then $E \preceq_s F$ iff $E \preceq F$;*
- *$\forall E, F \in \mathcal{E}(\mathcal{H})$ such that E or F is a projection: $E \preceq_s F$ iff $E \preceq F$.*

Let M_1 and M_2 be two spectral families.
Let $M_1 \vee_s M_2$ and $M_1 \wedge_s M_2$ be the maps from $\mathbb{R}$ to $\Pi(\mathcal{H})$ such that $\forall\lambda \in \mathbb{R}$:

- $(M_1 \vee_s M_2)(\lambda) := M_1(\lambda) \wedge M_2(\lambda)$;
- $(M_1 \wedge_s M_2)(\lambda) := \bigwedge_{\mu>\lambda}(M_1(\mu) \vee M_2(\mu))$.

Theorem

- *$M_1 \vee_s M_2$ and $M_1 \wedge_s M_2$ are spectral families.*

By $E \wedge_s F$ and $E \vee_s F$ we denote the effects whose spectral families are $(M^E \wedge_s M^F)$ and $(M^E \vee_s M^F)$, respectively.

THEOREM (OLSON 1971, H. F. DE GROOTE 2005)

- $\langle \mathcal{E}(\mathcal{H}), \wedge_s, \vee_s, \mathbb{O}, \mathbb{I} \rangle$ *is a bounded lattice such that* $\forall E, F \in \mathcal{E}(\mathcal{H})$: $E \preceq_s F$ *iff* $E = E \wedge_s F$.
- $\forall P, Q \in \Pi(\mathcal{H})$: $P \wedge Q = P \wedge_s Q$;
  $P \vee Q = P \vee_s Q$.

Let M be a spectral family. Let M' be the map from $\mathbb{R}$ to $\Pi(\mathcal{H})$ such that

$$\forall \lambda \in \mathbb{R}: \quad M'(\lambda) := \mathbb{I} - \bigvee_{\mu < 1 - \lambda} M(\mu).$$

For any effect E, the map $(M^E)'$ is the spectral family associated to $E' := \mathbb{I} - E$.

Theorem (H. F. de Groote 2005, A. L., F. Paoli, R. Giuntini 2016)

$\langle \mathcal{E}(\mathcal{H}), \wedge_s, \vee_s, {}', \mathbb{O}, \mathbb{I} \rangle$ *is a Kleene lattice, i.e., a bounded involution lattice that satisfies the (lattice-theoretic equivalent) Kleene condition:*

$$\forall E, F \in \mathcal{E}(\mathcal{H}) : \quad E \wedge_s E' \preceq_s F \vee_s F'.$$

(A possible contradiction always implies a possible truth)

In general: $E \vee_s E' \neq \mathbb{I}$.

What is the relationship between the orthomodular law(s) and the excluded middle law (noncontradiction) law?

Theorem

Let $\mathcal{L}$ be a bounded involution lattice. The following conditions are equivalent:

1. $\mathcal{L}$ *satisfies the orthomodular property:* $\forall a, b \in L: \quad \text{if } a \preceq b, \text{ then } b = a \vee (a' \wedge b)$;
2. $\mathcal{L}$ *satisfies the orthomodular equation:* $\forall a, b \in L: \quad a \vee b = ((a \vee b) \wedge b') \vee b$;
3. *the Sasaki implication* $\rightarrow_S$ $(a \rightarrow_S b := a' \vee (a \wedge b))$ *satisfies the modus ponens rule:* $(a \rightarrow_S b) \wedge a \preceq b$.

As a consequence:

if $\mathcal{L}$ is a bounded involution lattice that satisfies the orthomodular property

$$\Downarrow$$

$$\mathcal{L} \text{ is an orthé lattice} : \ \forall a \in L: \ a \vee a' = \mathbf{1}.$$

No room for "orthomodularity" in an unsharp universe?

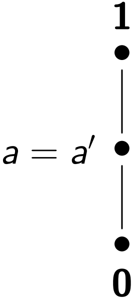
Recall that if $\mathcal{L}$ is an ortholattice , then $\mathcal{L}$ satisfies the orthomodularity law

iff

$\mathcal{L}$ satisfies the following condition (orthomodularity 4):

$$\forall a, b \in L : \text{ if } a \preceq b \text{ and } a' \wedge b = \mathbf{0}, \text{ then } a = b$$

- There are Kleene lattices that **satisfy** "orthomodularity 4" but **violates** the excluded middle law!



DEFINITION

A paraorthomodular lattice is a bounded involution Kleene lattice $\mathcal{L}$ that satisfies the paraorthomodular property (="orthomodularity 4"):

$$\forall a, b \in L: \text{ if } a \preceq b \text{ and } a' \wedge b = \mathbf{0}, \text{ then } a = b.$$

Theorem (A. L., F. Paoli, R. Giuntini 2016)

- $\langle \mathcal{E}(\mathcal{H}), \wedge_s, \vee_s, ', \mathbb{O}, \mathbb{I} \rangle$ *is a paraorthomodular lattice*

Theorem (A. L., F. Paoli, R. Giuntini 2016)

- *The class $\mathbb{PL}$ of all paraorthomodular lattices is a proper quasivariety;*
- $HSP(\mathbb{PL}) = \mathbb{KL}$;
- *If $\mathcal{L}$ is a modular Kleene lattice, then $\mathcal{L}$ is a paraorthomodular lattice.*

In the frame of Kleene lattices we recover well know results of orthomodular lattice theory:

Distributivity $\underset{\not\Leftarrow}{\Rightarrow}$ Modularity $\underset{\not\Leftarrow}{\Rightarrow}$ Paraorthomodularity

Theorem (Dvurecenskij, Gudder,...de Groote)

Let E be an effect. The following conditions are equivalent:

- *E is a projection;*
- $E \vee_s E' = \mathbb{I}$ *(E is "sharp").*

SHARP EFFECTS = PROJECTIONS

$$\forall E \in \mathcal{E}(\mathcal{H}) : \quad E^{\sim} := P_{ker(E)}$$

Theorem (The collapse of sharpness)

Let E be an effect. The following conditions are equivalent:

- *E is a projection;*
- $E \vee_s E' = \mathbb{I}$ *(E is "sharp");*
- $E' = E^{\sim}$.

THEOREM (H. F. DE GROOTE 2005, ET AL.)

The operation $^{\sim}$ *is a* *Brouwer complement*, *i.e.*, $\forall E, F \in \mathcal{E}(\mathcal{H})$:

- $E \preceq_s E^{\sim\sim}$;
- *if* $E \preceq_s F$, *then* $F^{\sim} \preceq_s E^{\sim}$;
  (equivalently: $(E \vee_s F)^{\sim} = E^{\sim} \wedge_s F^{\sim}$*)*
- $E \wedge_s E^{\sim} = \mathbb{O}$;
- $E^{\sim'} = E^{\sim\sim}$.

Is it possible to characterize in a **lattice-theoretic way** a satisfactory notion of "projection" in such a way that different notions of sharpness turns out to be equivalent ?

To do that we need two complements:

- a fuzzy-like complement ($'$);
- an intuitionistic-like complement ($\sim$).

DEFINITION

A Brouwer Zadeh lattice (BZ-lattice) is a structure $\mathcal{L} = \langle L\,, \wedge\,, \vee\,, {}'\,, {}^{\sim}\,, \mathbf{0}\,, \mathbf{1}\rangle$, where:

- $\langle L\,, \wedge\,, \vee\,, {}'\,, \mathbf{0}\,, \mathbf{1}\rangle$ is a Kleene lattice;
- $^{\sim}$ is an intuitionistic-like complement, i.e., $\forall a, b \in L$:
  - $a \preceq a^{\sim\sim}$;
  - $a \wedge a^{\sim} = \mathbf{0}$;
  - if $a \preceq b$ then $b^{\sim} \preceq a^{\sim}$;
    (equivalently, $(a \vee b)^{\sim} = a^{\sim} \wedge b^{\sim}$)
- $\forall a \in L$:
  - $a^{\sim\prime} = a^{\sim\sim}$ ($^{\sim}$ collapses into $'$ on $^{\sim}$-elements).

- The class of all BZ-lattices is a variety ($\mathbb{BZL}$).

Definition

- $\Diamond(a) := a^{\sim\sim}$;
- $\Box(a) = a'^{\sim}$.

DEFINITION

Let $\mathcal{L}$ be a BZ-lattice. An element $a \in L$ is said to be Kleene-sharp iff $a \vee a' = \mathbf{1}$.

$S_K(L)$ denotes the class of Kleene-sharp elements of $\mathcal{L}$.

- $S_K(L)$ is **closed** under the operations $'$ and $^\sim$, but **not** under the operations $\wedge$ or $\vee$.

Definition

An element $a \in L$ is said to be Brouwer-sharp iff $a \vee a^{\sim} = \mathbf{1}$.

$S_B(L)$ denotes the class of all Brouwer-sharp elements of $\mathcal{L}$.

- $S_B(L)$ is **closed** under the operations $'$ and $^{\sim}$, but **not** under the operations $\wedge$ or $\vee$.

DEFINITION

An element $a \in L$ is $\Diamond$-sharp iff $\Diamond(a) := a^{\sim\sim} = a$.

$S_\Diamond(L)$ denotes the class of all $\Diamond$-sharp elements of $\mathcal{L}$.

- $S_\Diamond(L)$ is **closed** under the operations $', ^\sim, \wedge, \vee$. Thus, $S_\Diamond(L)$ is the universe of a sub-BZ ortholattice of $\mathcal{L}$.

Theorem (G. Cattaneo and G. Marino 1994, H. F. de Groote 2005)

If $\langle \mathcal{E}(\mathcal{H}), \wedge_s, \vee_s, ', \sim, \mathbb{O}, \mathbb{I} \rangle$ *is the BZ-lattice of effects, then*

$$S_{\Diamond}(\mathcal{E}(\mathcal{H})) = S_B(\mathcal{E}(\mathcal{H})) = S_K(\mathcal{E}(\mathcal{H})).$$

Theorem (G. Cattaneo and G. Marino 1984)

If $\mathcal{L} = \langle L\,,\,\wedge\,,\vee\,,\,'\,,\,^{\sim}\,,\,\mathbf{0}\,,\mathbf{1}\rangle$ *is a BZ-lattice, then*

$$S_{\Diamond}(L) \subset S_B(L) \subset S_K(L).$$

Definition

A BZ*-lattice is a BZ-lattice $\mathcal{L}$ that satisfies the following condition:

$$\forall a \in L: \quad (a \wedge a')^{\sim} \preceq a^{\sim} \vee a'^{\sim}.$$

Possible reading: the strong negation of a possible contradiction $(a \wedge a')$ implies that either a is necessary or a is impossible. In a modal tone of voice:

$$(\Diamond(a \wedge a'))' \preceq \Box(a) \vee (\Diamond(a))'.$$

How to recover the collapse of sharpness in an abstract way: a first attempt

Theorem (A. L., F. Paoli, R. Giuntini 2016)

$\langle \mathcal{E}(\mathcal{H}), \wedge_s, \vee_s, ', \sim, \mathbb{O}, \mathbb{I} \rangle$ *is a BZ^*-lattice.*

Theorem (A. L., F. Paoli, R. Giuntini 2016)

If $\mathcal{L}$ is a BZ^*-lattice, then

$$S_B(L) = S_K(L).$$

However, in general,

$$S_K(L) \not\subseteq S_\Diamond(L).$$

Definition

A paraorthomodular BZ-lattice (PBZ-lattice) is a BZ-lattice such that the reduct $\langle L, \wedge, \vee, ', \mathbf{0}, \mathbf{1}\rangle$ is paraorthomodular (if $a \leq b$ and $a' \wedge b = \mathbf{0}$, then $a = b$).

Theorem (A. L., F. Paoli, R. Giuntini 2016)

Let $\mathcal{L}$ be a BZ-lattice. The following conditions are equivalent:*

- $\mathcal{L}$ is *paraorthomodular*:
  $\forall a, b \in L$: *if* $a \preceq b$ *and* $a' \wedge b = \mathbf{0}$, *then* $a = b$.
- $\forall a, b \in L$:
  $(\Diamond(a) \rightarrow_S \Diamond(b)) \wedge \Diamond(a) \preceq \Diamond(b)$, *i.e.*

  $(a^{\sim} \vee (a^{\sim\sim} \wedge b^{\sim\sim})) \wedge a^{\sim\sim} \leq b^{\sim\sim}$
  (*$\Diamond$-orthomodularity* or *$\Diamond$-modus ponens*).

$\Downarrow$

Unlike paraorthomodular Kleene lattices, the class of all paraorthomodular BZ*-lattices is a variety (denoted $\mathbb{PBZL}^*$)

Theorem (A. L., F. Paoli, R. Giuntini 2016)

$\langle \mathcal{E}(\mathcal{H}), \wedge_s, \vee_s, ', \sim, \mathbb{O}, \mathbb{I} \rangle$ *is a PBZ*-lattice.*

Theorem (A. L., F. Paoli, R. Giuntini 2016)

If $\mathcal{L}$ is a PBZ-lattice, then*

- $S_B(L) = S_K(L) = S_\Diamond(L)$.
- *The sub-ortholattice* $\langle S_K(L), \wedge, \vee, ', \mathbf{0}, \mathbf{1} \rangle$ *is orthomodular.*

Theorem (A. L., F. Paoli, R. Giuntini 2016)

$\langle \mathcal{E}(\mathcal{H}), \wedge_s, \vee_s, ', \sim, \mathbb{O}, \mathbb{I} \rangle$ *is a PBZ*-lattice.*

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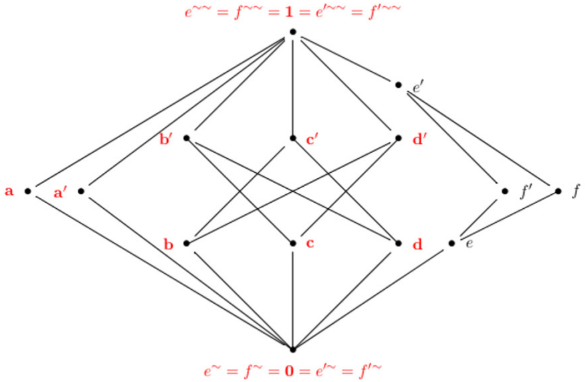

The smallest **nonmodular** PBZ*-lattice. The red lattice is the orthomodular lattice $\mathcal{L}_{10}$, i.e., the smallest **nonmodular** orthomodular lattice

PBZ*-lattices are faithful generalizations of $\mathcal{E}(\mathcal{H})$ with a good "orthomodular flavor"!

Thank you!

Thank you!

- Determine whether there is an equation in the $^{\sim}$-free fragment which holds in the class of PBZ*-lattices of effects of some Hilbert space but fails in $\mathbb{PBZL}^*$.
- Develop a theory of commuting elements and a notion of commutator for PBZ*-lattices, verifying if it is possible to establish some analogue of the Foulis-Holland theorem for orthomodular lattices.

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