The strong version of a sentential logic

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Introduction

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Modal Logic: Given a class of Kripke models we have the local consequence relation and the global consequence relation. The first is equivalential and the second algebraizable.

Substructural logics: Given a variety of commutative integral residuated lattices, we have the 1-assertional logic and the logic preserving degrees of truth (defined by the order of the lattices). The first is algebraizable, and the second can be non-protoalgebraic.

Subintuitionistic logics: Like in modal logic, given a class of Kripke models we have the local and the global consequence relation. Depending on the class of Kripke models they are protoalgebraic or not. If we take the class of all Kripke models both are non-protoalgebraic.

In

J.M. Font and R. J. *The strong version of a protoalgebraic logic*, Arch. Math. Logic 40 (2001),

we developed

a framework to account for the mentioned phenomena,

in the setting of abstract algebraic logic, but only for protoalgebraic logics.

The main tool to introduce the concept of the strong version of a protoalgebraic logic $\mathcal{S}$ was the notion of Leibniz $\mathcal{S}$-filter.

Now we have extended the theory to any logic and we have the concept of the strong version of an arbitrary given logic.

The main tool is a new notion of Leibniz $\mathcal{S}$-filter, this time defined for every logic $\mathcal{S}$. It is introduced in

> H. Albuquerque, J.M. Font and R. J. *Compatibility operators in abstract algebraic logic*, JSL 81 (2016).

The notion, although different from the one given for protoalgebraic logics, coincides in extension with it, when restricted to the logics of this type.

Preliminary basic concepts

Let $\mathcal{S}$ be a logic, understood as a consequence relation $\vdash_{\mathcal{S}}$ (invariant under substitutions) over the formula algebra with denumerably many variables $(x, y, z, \ldots)$ and in a propositional language $L_{\mathcal{S}}$.

Let $\boldsymbol{A}$ be an algebra of type $L_{\mathcal{S}}$.

A set $F \subseteq A$ is an $\mathcal{S}$-filter if it is closed under the interpretations of the pairs (Γ, φ) such that $\Gamma \vdash_{\mathcal{S}} \varphi$. The set (complete lattice) of the $\mathcal{S}$-filters of $\boldsymbol{A}$ is denoted by $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.

Let $F \subseteq A$. The Leibniz congruence of F is the largest congruence θ of $\boldsymbol{A}$ compatible with F (i.e. such that F is a union of equivalence classes of θ). It is denoted by $\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$.

The Suszko $\mathcal{S}$-congruence of F, denoted $\widetilde{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}(F)$, is the intersection of the Leibniz congruences of all the $\mathcal{S}$-filters of $\boldsymbol{A}$ that include F.

The algebraic counterpart of $\mathcal{S}$ is the class of algebras

$$\mathrm{Alg}\mathcal{S} = \{\boldsymbol{A} : \exists F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \text{ s.t. } \widetilde{\Omega}^{\boldsymbol{A}}_{\mathcal{S}}(F) \text{ is the identity}\}$$

The class of algebras

$$\mathrm{Alg}^{*}\mathcal{S} = \{\boldsymbol{A} : \exists F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} \text{ s.t. } \Omega^{\boldsymbol{A}}(F) \text{ is the identity}\}$$

is also important in abstract algebraic logic.
It turns out that $\mathrm{Alg}\mathcal{S}$ is the closure of $\mathrm{Alg}\mathcal{S}$ under subdirect products.
For protoalgebraic logics, $\mathrm{Alg}^{*}\mathcal{S} = \mathrm{Alg}\mathcal{S}$.

Let $\boldsymbol{A}$ be an algebra of type $L_{\mathcal{S}}$. Let $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. The set

$$[\![F]\!]^{*}_{\mathcal{S}} := \{G \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A} : \boldsymbol{\Omega}^{\boldsymbol{A}}(F) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(G)\}$$

has a least element, that we denote by F^{*}.

Definition

F is a Leibniz $\mathcal{S}$-filter if it is the least element of its set $[\![F]\!]^{*}_{\mathcal{S}}$, that is, if $F^{*} = F$.

- $\mathcal{F}i^{*}_{\mathcal{S}}\boldsymbol{A}$ denotes the set of the Leibniz $\mathcal{S}$-filters of $\boldsymbol{A}$.

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- $\mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$ denotes the set of the Leibniz $\mathcal{S}$-filters of $\boldsymbol{A}$.
- Let $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$. The following are equivalent:
  - F is a Leibniz $\mathcal{S}$-filter of $\boldsymbol{A}$,
  - $F/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$ is the least $\mathcal{S}$-filter of $\boldsymbol{A}/\boldsymbol{\Omega}^{\boldsymbol{A}}(F)$.
- Let $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, then $(F^*)^* = F^*$ and therefore F^* is Leibniz.

The strong version of a logic

Definition

The strong version of a logic $\mathcal{S}$ is the logic $\mathcal{S}^+$ given by the class of matrices

$$\{\langle \boldsymbol{A}, F\rangle : \boldsymbol{A} \text{ is an } L_{\mathcal{S}}\text{-algebra and } F \in \mathcal{F}i^{*}_{\mathcal{S}}\boldsymbol{A}\}.$$

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Both, in the definition and in the characterization we can restrict the algebras to the members of $\mathrm{Alg}\mathcal{S}$ (and also of $\mathrm{Alg}^{*}\mathcal{S}$).

Some facts

• $\mathcal{S}^+$ is an extension of $\mathcal{S}$.

• If $\mathcal{S}$ does not have theorems, then $\mathcal{S}^+$ is the almost inconsistent logic (whose only theories are $\emptyset$ and Fm).

• The Leibniz $\mathcal{S}$-filters are $\mathcal{S}^+$-filters. Hence, $\mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A} \subseteq \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} \subseteq \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, for every $\boldsymbol{A}$.

• $\mathcal{S}$ and $\mathcal{S}^+$ have the same theorems. More generally, for every $\boldsymbol{A}$ the least $\mathcal{S}$-filter and the least $\mathcal{S}^+$-filter coincide.

• $\mathcal{S}^+$ is the largest of all the logics $\mathcal{S}'$ with the property that for every algebra the Leibniz $\mathcal{S}$-filters are $\mathcal{S}'$-filters.

• If $\mathcal{S} \leq \mathcal{S}' \leq \mathcal{S}^+$, then $\mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{S}'}\boldsymbol{A}$, for every $\boldsymbol{A}$ and hence $(\mathcal{S}')^+ = \mathcal{S}^+$.

In particular, $\mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{S}^+}\boldsymbol{A}$ and $(\mathcal{S}^+)^+ = \mathcal{S}^+$.

In between $\mathcal{S}$ and $\mathcal{S}^+$ there can be many logics $\mathcal{S}'$. In fact, in some cases a continuum of them.

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In between $\mathcal{S}$ and $\mathcal{S}^+$ there can be many logics $\mathcal{S}'$. In fact, in some cases a continuum of them.

• All the $\mathcal{S}$-filters of $\mathcal{S}$ are Leibniz if and only if for every $\boldsymbol{A}$, $\boldsymbol{\Omega}^{\boldsymbol{A}}(.)$ is order reflection on $\mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$.

• If $\mathcal{S}$ is truth-equational, then all is $\mathcal{S}$-filters are Leibniz and therefore $\mathcal{S} = \mathcal{S}^+$.

- It is not always the case that $\mathcal{F}i^{*}_{\mathcal{S}}\boldsymbol{A} = \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$.

For example, if $\mathcal{S}$ does not have theorems, then $\mathcal{F}i^{*}_{\mathcal{S}}\boldsymbol{A} \subsetneq \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$.

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- We will study conditions that imply that $\mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A} = \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$.

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The following conditions are equivalent.

- $\mathcal{F}i^{*}_{\mathcal{S}}\boldsymbol{A} = \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$, for every $\boldsymbol{A}$,
- $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is order reflecting over $\mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$, for every $\boldsymbol{A}$.

Thus, when $\mathcal{S}^+$ is truth-equational, $\mathcal{F}i^{*}_{\mathcal{S}}\boldsymbol{A} = \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A}$, for every $\boldsymbol{A}$.

Equational definability

Definition

We say that $\mathcal{S}$ has its Leibniz filters equationally definable if there exists a set of equations $\tau(x)$ in one variable such that for every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$,

$$F^* = \{a \in A : \tau^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)\},$$

where $\tau^{\boldsymbol{A}}(a) = \{\langle \varepsilon^{\boldsymbol{A}}(a), \delta^{\boldsymbol{A}}(a)\rangle : \varepsilon \approx \delta \in \tau(x)\}$.

• If $\mathcal{S}$ has its Leibniz filters equationally definable by $\tau(x)$, then for every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$,

$$F \text{ is a Leibniz } \mathcal{S}\text{-filter} \quad \text{iff} \quad F = \{a \in A : \tau^{\boldsymbol{A}}(a) \subseteq \boldsymbol{\Omega}^{\boldsymbol{A}}(F)\}.$$

• The following are equivalent:

- $\mathcal{S}$ has its Leibniz filters equationally definable by $\tau(x)$.
- $\tau\boldsymbol{A} := \{a \in A : \boldsymbol{A} \models \tau(x)[a]\}$ is the least $\mathcal{S}$-filter of $\boldsymbol{A}$, for every $\boldsymbol{A} \in \mathrm{Alg}\mathcal{S}$.

- If $\mathcal{S}$ has its Leibniz filters equationally definable by $\tau(x)$, then
  - $\mathcal{S}^+$ is the τ-assertional logic of Alg$\mathcal{S}$.
  - $\mathcal{S}^+$ is truth-equational (with τ as a set of defining equations).
  - $\mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$, for every $\boldsymbol{A}$.

Logical definability

Definition

We say that $\mathcal{S}$ has its Leibniz filters logically definable if there exists a set of rules $\mathcal{H} = \{\Gamma_i \vdash \varphi_i : i \in I\}$ such that for every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, F is a Leibniz $\mathcal{S}$-filter if and only if it is closed under the interpretation of every rule in $\mathcal{H}$.

If $\mathcal{S}$ has its Leibniz filters logically definable, then for every $\boldsymbol{A}$, $\mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$.

If $\mathcal{S}$ has its Leibniz filters logically definable by a set of rules $\mathcal{H}$, then $\mathcal{S}^+$ is the extension of $\mathcal{S}$ given by the rules in $\mathcal{H}$.

Explicit definability

Definition

A logic $\mathcal{S}$ has its Leibniz filters explicitly definable if there exists a set of formulas $\Gamma(x)$ in one variable x such that for every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$,

$$F^* = \{a \in A : \Gamma^{\boldsymbol{A}}(a) \subseteq F\}.$$

If $\mathcal{S}$ has its Leibniz filters explicitly definable by $\Gamma(x)$, then for every $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$,

$$F \text{ is a Leibniz } \mathcal{S}\text{-filter} \quad \text{iff} \quad F = \{a \in A : \Gamma^{\boldsymbol{A}}(a) \subseteq F\}.$$

If $\mathcal{S}$ has its Leibniz filters explicitly definable by $\Gamma(x)$, then

- $\mathcal{S}$ has its Leibniz filters logically definable by the set of rules $\{x \vdash \varphi : \varphi \in \Gamma(x)\}$,
- for every $\boldsymbol{A}$, $\mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}$,
- $\mathcal{S}^+$ is the extension of $\mathcal{S}$ given by the rules in $\{x \vdash \varphi : \varphi \in \Gamma(x)\}$.

Let $\mathcal{S}$ have its Leibniz filters explicitly definable by $\Gamma(x)$. Then for all $\Delta \cup \{\varphi\} \subseteq Fm$,

$$\Delta \vdash_{\mathcal{S}^+} \varphi \quad \iff \quad \Gamma(\Delta) \vdash_{\mathcal{S}} \varphi.$$

Moreover,

1. $\Gamma(x) \vdash_{\mathcal{S}} x$.
2. $\Gamma(x) \dashv\vdash_{\mathcal{S}^+} x$.
3. $\Gamma\big(\Gamma(x)\big) \dashv\vdash_{\mathcal{S}} \Gamma(x)$.
4. If $\Delta \vdash_{\mathcal{S}} \varphi$, then $\Gamma(\Delta) \vdash_{\mathcal{S}} \Gamma(\varphi)$, for all $\Delta \cup \{\varphi\} \subseteq Fm$.

Like the behaviour of the set $\{\Box^n x : n \in \omega\}$ in the local consequence of the class of all Kripke models.

A logic $\mathcal{S}$ has its Leibniz filters explicitly definable if and only if there is a set of formulas $\Gamma(x)$ such that

1. $\mathcal{S}$ has its Leibniz filters logically definable by the set of rules $x \vdash \Gamma(x)$,
2. for all $\Delta \cup \{\varphi\} \subseteq Fm$ such that $\Delta \vdash_{\mathcal{S}} \varphi$ it holds that $\Gamma(\Delta) \vdash_{\mathcal{S}} \Gamma(\varphi)$

and moreover for every $\boldsymbol{A}$ and all $F \in \mathcal{F}i_{\mathcal{S}}\boldsymbol{A}$, F^* is the largest Leibniz $\mathcal{S}$-filter included in F.

Positive modal logic $\mathcal{PML}$.

It is the negation-less and implication-less fragment of the local consequence relation of the class of all Kripke frames (with $\Diamond, \Box, \wedge, \vee, \top$, and $\bot$ as its language primitive symbols).

The class Alg$\mathcal{PML}$ is the variety PMA of positive modal algebras (M. Dunn). And $\mathcal{PML}$ is the logic of the order of PMA.

A positive modal algebra is an algebra $\boldsymbol{A} = \langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, \Box^{\boldsymbol{A}}, \Diamond^{\boldsymbol{A}}, \top^{\boldsymbol{A}}, \bot^{\boldsymbol{A}} \rangle$ where $\langle A, \wedge^{\boldsymbol{A}}, \vee^{\boldsymbol{A}}, 1, 0 \rangle$ is a bounded distributive lattice and for every $a, b \in A$:

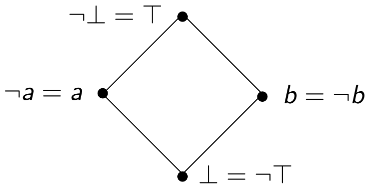
1. $\Box^{\boldsymbol{A}}(a \wedge^{\boldsymbol{A}} b) = \Box^{\boldsymbol{A}} a \wedge^{\boldsymbol{A}} \Box^{\boldsymbol{A}} b$
2. $\Diamond^{\boldsymbol{A}}(a \vee^{\boldsymbol{A}} b) = \Diamond^{\boldsymbol{A}} a \vee^{\boldsymbol{A}} \Diamond^{\boldsymbol{A}} b$
3. $\Box^{\boldsymbol{A}} a \wedge^{\boldsymbol{A}} \Diamond^{\boldsymbol{A}} b \leq \Diamond^{\boldsymbol{A}}(a \wedge^{\boldsymbol{A}} b)$
4. $\Box^{\boldsymbol{A}}(a \vee^{\boldsymbol{A}} b) \leq \Box^{\boldsymbol{A}} a \vee^{\boldsymbol{A}} \Diamond^{\boldsymbol{A}} b$
5. $\Box^{\boldsymbol{A}} \top^{\boldsymbol{A}} = \top^{\boldsymbol{A}}$
6. $\Diamond^{\boldsymbol{A}} \bot^{\boldsymbol{A}} = \bot^{\boldsymbol{A}}$

Alg$\mathcal{PML}$ and is different from Alg$^*\mathcal{PML}$.

Belnap-Dunn logic $\mathcal{B}$.

We take it in the language $\wedge, \vee, \neg, \bot, \top$.

Belnap-Dunn's logic $\mathcal{B}$ is the logic of the order of the variety DMA of De Morgan algebras, which is generated by the four-element De Morgan algebra



$\mathcal{PML}$	Belnap-Dunn: $\mathcal{B}$
fully selfextensional	idem
not protoalgebraic	idem
not truth-equational	idem
not Fregean	idem
$\mathcal{F}i_{\mathcal{PML}}\boldsymbol{A}$ = lattice filters	$\mathcal{F}i_{\mathcal{B}}\boldsymbol{A}$ = lattice filters
Leibniz filters eq. definable by $x \approx \top$	Leibniz filters eq. definable by $x \approx \top$
Leibniz filters explicitly definable by $\{\Box^n x : n \in \omega\}$	Leibniz filters not explicitly definable
Leibniz filters logically definable by (N) $x \vdash \Box x$	Leibniz filters logically definable by (DS) $x, \neg x \vee y \vdash y$
Leibniz filters = open filters	Leibniz filters = lattice filters closed by $\neg x \vee y$
$\mathcal{F}i_{\mathcal{PML}^+}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{PML}}\boldsymbol{A}$	$\mathcal{F}i_{\mathcal{B}^+}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{B}}\boldsymbol{A}$

$\mathcal{PML}$	Belnap-Dunn: $\mathcal{B}$
$\mathsf{Alg}\mathcal{PML}^+ \subsetneq \mathsf{Alg}\mathcal{PML} = \mathsf{PMA}$	$\mathsf{Alg}\mathcal{B}^+ = \mathsf{Alg}\mathcal{B} = \mathsf{DMA}$
$\mathcal{PML}^+ = \top$-assertional logic of PMA	$\mathcal{B}^+ = \top$-assertional logic of DMA
$\mathcal{PML}^+ = \mathcal{PML} \quad + \quad x \vdash \Box x$	$\mathcal{B}^+ = \mathcal{B} \quad + \quad x, \neg x \vee y \vdash y$
$\mathcal{PML}^+$ is truth-equational	idem
$\mathcal{PML}^+$ is not protoalgebraic	idem
$\mathcal{PML}^+$ is not selfextensional	idem

Substructural logics: the integral case

Let K be a variety of commutative and integral residuated lattices, in the language $\{\vee, \wedge, \odot, \to, 1\}$. Consider the logic $\mathcal{S}_{\mathsf{K}}^{\leq}$ of degrees of truth of K and its 1-assertional logic $\mathcal{S}_{\mathsf{K}}^{1}$, which is known to be BP-algebraizable. The $\mathcal{S}_{\mathsf{K}}^{1}$-filters on algebras in K are the implicative filters.

Substructural logics: the integral case

Let K be a variety of commutative and integral residuated lattices, in the language $\{\vee, \wedge, \odot, \rightarrow, 1\}$. Consider the logic $\mathcal{S}_{\mathsf{K}}^{\leq}$ of degrees of truth of K and its 1-assertional logic $\mathcal{S}_{\mathsf{K}}^{1}$, which is known to be BP-algebraizable. The $\mathcal{S}_{\mathsf{K}}^{1}$-filters on algebras in K are the implicative filters.

- $\mathcal{S}_{\mathsf{K}}^{1}$ is the strong version of $\mathcal{S}_{\mathsf{K}}^{\leq}$ (i.e. $\mathcal{S}_{\mathsf{K}}^{1} = (\mathcal{S}_{\mathsf{K}}^{\leq})^{+}$).

Substructural logics: the non-integral case

Let K be a variety of commutative residuated lattices (not necessarily integral). The usual substructural logic associated with K is the $\{1 \leq x\}$-assertional logic of K.

We denote it by $\mathcal{S}_{\mathsf{K}}^{\tau}$ (for $\tau = \{x \wedge 1 \approx 1\}$).

The logic $\mathcal{S}_{\mathsf{K}}^{\tau}$ is:

- BP-algebraizable with $\mathrm{Alg}\mathcal{S}_{\mathsf{K}}^{\tau} = \mathsf{K}$.
- The $\mathcal{S}_{\mathsf{K}}^{\tau}$-filters of any $\boldsymbol{A} \in \mathsf{K}$ are the implicative filters (i.e. the lattice filters closed under $\rightarrow$) that contain 1.

The logic $\mathcal{S}_{\mathsf{K}}^{\leq}$ of degrees of truth of K may not have theorems. If this is the case, its strong version is the almost inconsistent logic and different from $\mathcal{S}_{\mathsf{K}}^{\tau}$.

Let $\mathcal{S}_{\mathsf{K}}^{\preccurlyeq}$ be the least logic $\mathcal{S}$ such that $\mathcal{S}_{\mathsf{K}}^{\leq} \leq \mathcal{S} \leq \mathcal{S}_{\mathsf{K}}^{\tau}$ and with the same theorems as $\mathcal{S}_{\mathsf{K}}^{\tau}$. This logic can be defined as the logic of the class of matrices

$$\{\langle \boldsymbol{A}, [1 \wedge a) \rangle : \boldsymbol{A} \in \mathsf{K}, a \in A\}$$

and as the one of the class of matrices

$$\{\langle \boldsymbol{A}, F \rangle : \boldsymbol{A} \in \mathsf{K}, F \text{ is a lattice filter of } \boldsymbol{A} \text{ and } 1 \in F\}.$$

- $\mathsf{Alg}\mathcal{S}_{\mathsf{K}}^{\preccurlyeq} = \mathsf{K} = \mathsf{Alg}\mathcal{S}_{\mathsf{K}}^{\tau}$.

$\mathcal{S}_{\mathsf{K}}^{\leq}$, K $\subseteq$ CRIL variety	$\mathcal{S}_{\mathsf{K}}^{\preccurlyeq}$, K $\subseteq$ CRL non-integral variety
fully selfextensional	$\mathcal{S}_{\mathsf{K}}^{\preccurlyeq}$ is not selfextensional
$\mathcal{S}_{\mathsf{K}}^{\leq}$ is (fully) Fregean iff it is truth-equational iff it is algebraizable iff $\mathcal{S}_{\mathsf{K}}^{\leq} = (\mathcal{S}_{\mathsf{K}}^{\leq})^{+}$	$\mathcal{S}_{\mathsf{K}}^{\preccurlyeq}$ is truth-equational iff it is algebraizable iff $\mathsf{K} \models x \wedge (x \to y) \wedge 1 \leq y$ iff $\mathcal{S}_{\mathsf{K}}^{\preccurlyeq} = (\mathcal{S}_{\mathsf{K}}^{\preccurlyeq})^{+}$
$\mathcal{S}_{\mathsf{K}}^{\leq}$ is protoalgebraic iff there exists $n \in \omega$ such that $\mathsf{K} \models x \wedge (x \to y)^{n} \leq y$	? ?
$\mathcal{F}i_{\mathcal{S}_{\mathsf{K}}^{\leq}} \boldsymbol{A}$ = lattice filters	$\mathcal{F}i_{\mathcal{S}_{\mathsf{K}}^{\preccurlyeq}} \boldsymbol{A}$ = lattice filters with 1
Leibniz filters eq. definable by $x \approx \top$	Leibniz filters eq. definable by $1 \leq x$
Leibniz filters explicitly definable iff it is protoalgebraic (definable by $\{x^{n} : n \in \omega\}$)	If the Leibniz filters are explicitly definable, then it is protoalgebraic
Leibniz filters logically definable by Modus Ponens	Leibniz filters logically definable by Modus Ponens
Leibniz filters = implicative filters	Leibniz filters = implicative filters that contain 1

$\mathcal{S}^{\leq}_{\mathsf{K}}$, K $\subseteq$ CRIL variety	$\mathcal{S}^{\preccurlyeq}_{\mathsf{K}}$, K $\subseteq$ CRL non-integral variety
$\mathcal{F}i_{(\mathcal{S}^{\preccurlyeq}_{\mathsf{K}})^+}\boldsymbol{A} = \mathcal{F}i^{*}_{\mathcal{S}^{\preccurlyeq}_{\mathsf{K}}}\boldsymbol{A}$	$\mathcal{F}i_{\mathcal{B}^+}\boldsymbol{A} = \mathcal{F}i^{*}_{\mathcal{B}}\boldsymbol{A}$
$\mathsf{Alg}(\mathcal{S}^{\leq}_{\mathsf{K}})^+ = \mathsf{Alg}\mathcal{S}^{\leq}_{\mathsf{K}} = \mathsf{K}$	$\mathsf{Alg}(\mathcal{S}^{\leq}_{\mathsf{K}})^+ = \mathsf{Alg}\mathcal{S}^{\preccurlyeq}_{\mathsf{K}} = \mathsf{K}$
$(\mathcal{S}^{\leq}_{\mathsf{K}})^+ =$ 1-assertional logic of K	$(\mathcal{S}^{\preccurlyeq}_{\mathsf{K}})^+ = \{1 \leq x\}$-assertional logic of K
$(\mathcal{S}^{\leq}_{\mathsf{K}})^+ = \mathcal{S}^{\leq}_{\mathsf{K}} + (MP)$	$(\mathcal{S}^{\preccurlyeq}_{\mathsf{K}})^+ = \mathcal{S}^{\preccurlyeq}_{\mathsf{K}} + (MP)$
$(\mathcal{S}^{\leq}_{\mathsf{K}})^+$ is BP-algebraizable	$(\mathcal{S}^{\preccurlyeq}_{\mathsf{K}})^+$ is BP-algebraizable
$(\mathcal{S}^{\leq}_{\mathsf{K}})^+$ is selfextensional iff $\mathcal{S}^{\leq}_{\mathsf{K}} = (\mathcal{S}^{\leq}_{\mathsf{K}})^+$	$(\mathcal{S}^{\preccurlyeq}_{\mathsf{K}})^+$ is not selfextensional

Question:

Is there an interesting propertey Φ such that

$$\mathcal{S} \text{ has } \Phi \quad \text{iff} \quad \text{for every } \boldsymbol{A},\ \mathcal{F}i_{\mathcal{S}^+}\boldsymbol{A} = \mathcal{F}i^*_{\mathcal{S}}\boldsymbol{A}\ ?$$