Embedding Ext**IPC** into Ext**PLL** via canonical formulas

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Introduction/Notation

- $\mathcal{L}_{\mathbf{IPC}}$ denote the language of propositional logic.
- **IPC** denotes the intuitionistic propositional calculus.
- Ext**IPC** denotes the lattice of all superintuitionistic logics (si-logics).
- An intuitionistic modal logic is a collection of formulas in the language $\mathcal{L}_{\mathbf{IPC}} \cup \{\bigcirc\}$, closed under MP and substitution.

Propositional lax logic **PLL**

Definition

PLL is an intuitionistic modal logic with a peculiar modality $\bigcirc$ that is axiomatized by

- $p \to \bigcirc p$
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- Different semantics were studied by Goldblatt in [Gol81], by Dragalin in [Dra88] and in [FM97].
- **PLL** has the finite model property and is decidable [Gol81], [FM97], [WZ98].

Nuclear Heyting algebras

Definition

Let A be a Heyting algebra. A *nucleus* on A is a function $j : A \to A$ such that for all $a, b \in A$

$$a \leq j(a), \quad j(j(a)) = j(a), \quad j(a \wedge b) = j(a) \wedge j(b).$$

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Theorem (Gol81)

Every $M \in \mathrm{Ext}\mathbf{PLL}$ *is sound and complete with respect to its corresponding variety of nuclear Heyting algebras.*

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- Wolter and Zakharyaschev studied such preservation results by embedding intuitionistic modal logics into classical bi-modal logics.

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- Let A be a finite s.i. Heyting algebra, $D \subseteq A^2$. Then the canonical formula $\beta(A, D)$ encodes
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  - the $(\wedge, \to, 0)$- structure of A fully and
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- Every formula in $\mathcal{L}_{\mathbf{IPC}}$ is equivalent to a finite conjunction of canonical formulas.
- Thus, all si-logics are axiomatizable by canonical formulas.

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$$\begin{aligned}\beta(\mathfrak{A}, D^{\vee}, D^{\bigcirc}) := & \bigwedge\{p_{a*b} \leftrightarrow (p_a * p_b) \mid a, b \in A, * \in \{\wedge, \rightarrow\}\} \wedge \{p_0 \leftrightarrow 0\} \\ & \bigwedge\{p_{a\vee b} \leftrightarrow (p_a \vee p_b) \mid a, b \in D^{\vee}\} \wedge \\ & \bigwedge\{\bigcirc p_a \rightarrow p_{j(a)} \mid a \in A\} \wedge \\ & \bigwedge\{p_{j(a)} \rightarrow \bigcirc p_a \mid a \in D^{\bigcirc}\} \\ & \longrightarrow p_s.\end{aligned}$$

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$\beta(\mathfrak{A}, D^{\vee}, D^{\bigcirc})$ is called the canonical formula of $(\mathfrak{A}, D^{\vee}, D^{\bigcirc})$.

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  - If in addition,
    - $f(a \vee b) = f(a) \vee f(b)$ for every $(a, b) \in D^{\vee}$ and
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Theorem

For every nuclear Heyting algebra $\mathfrak{B} = (B, j)$, the following are equivalent:

1. $\mathfrak{B} \not\models \beta(\mathfrak{A}, D^{\vee}, D^{\bigcirc})$.
2. *There is a homomorphic image $\mathfrak{C}$ of $\mathfrak{B}$ and a $(D^{\vee}, D^{\bigcirc})$-stable embedding from $\mathfrak{A}$ into $\mathfrak{C}$.*

Axiomatic completeness

Proposition

For every **PLL***-formula* φ*, there is a finite collection* $\{(\mathfrak{A}_i, D_i^{\vee}, D_i^{\bigcirc})\}_{1\leq i\leq n}$ *such that for each nuclear Heyting algebra* $\mathfrak{B}$*, TFAE:*

1. $\mathfrak{B} \not\models \varphi$.
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Corollary

1. *Every formula in the language of* **PLL** *is equivalent to a finite conjunction of canonical formulas.*
2. *Every* $M \in \mathrm{Ext}\mathbf{PLL}$ *can be axiomatized by canonical formulas.*

Comparison to Zakharyschev's canonical formulas

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- By "deleting the parts with $\bigcirc$", we obtain an algebraic version of Zakharyaschev.'s canonical formulas:
- For a finite s.i. Heyting algebra A let

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- For every Heyting algebra $\mathfrak{B}$, TFAE:
  1. $B \not\models \beta(A, D^{\vee})$.
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Lemma

Let $\mathfrak{A} = (A, j)$ *be finite and s.i.,* $D^{\vee} \subseteq A^2$, $D^{\bigcirc} \subseteq A$*. Let* $L = \mathbf{IPC} + \Gamma$*. Then*

$$L \vdash \beta(A, D^{\vee}) \text{ implies } \mathbf{PLL} + \Gamma \vdash \beta(\mathfrak{A}, D^{\vee}, D^{\bigcirc}).$$

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- $B \not\models \beta(A, D^\vee)$. Since $B \models L$, $\beta(A, D^\vee) \notin L$.

□

Theorem

Let $L = \mathbf{IPC} + \Gamma$ be a si-logic. If L has one of the properties

- *tabularity,*
- *the fmp,*
- *Kripke completeness,*
- *decidability and Kripke completeness,*

then $\mathbf{PLL} + \Gamma$ also enjoys the same property.

Sketch: fmp is preserved.

- Suppose $\mathbf{PLL} + \Gamma \nvdash \beta(\mathfrak{A}, \mathrm{D}^{\vee}, \mathrm{D}^{\bigcirc})$. Then $L \nvdash \beta(A, \mathrm{D}^{\vee})$.

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- Since C is an L-algebra, $\mathfrak{C}$ validates Γ and is finite.

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- Thank you!