Finitely Protoalgebraic and Finitely Weakly Algebraizable Logics

Tuomas Hakoniemi

Universitat de Barcelona

SYSMICS2016

Logics

Let **Fm** be the formula algebra in some language over a countably infinite set Var of variables.

Definition

A logic is a pair $\mathcal{L} = \langle \mathbf{Fm}, \vdash_{\mathcal{L}} \rangle$, where **Fm** is the formula algebra and $\vdash_{\mathcal{L}}$ is a relation between sets of formulas and single formulas satisfying the following:

(i) if $\varphi \in \Gamma$, then $\Gamma \vdash_{\mathcal{L}} \varphi$; (reflexivity)

(ii) if $\Gamma \vdash_{\mathcal{L}} \varphi$ and $\Delta \vdash_{\mathcal{L}} \gamma$ for all $\gamma \in \Gamma$, then $\Delta \vdash_{\mathcal{L}} \varphi$; (cut)

(iii) if $\Gamma \vdash_{\mathcal{L}} \varphi$ and $\sigma \in \mathrm{End}(\mathbf{Fm})$, then $\sigma\Gamma \vdash_{\mathcal{L}} \sigma\varphi$. (structurality)

A logic is finitary if moreover the following holds

(iv) if $\Gamma \vdash_{\mathcal{L}} \varphi$, then there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathcal{L}} \varphi$.

A set T of formulas is an $\mathcal{L}$-theory if it is closed under the consequence relation of $\mathcal{L}$, i.e. if $T \vdash_{\mathcal{L}} \varphi$ implies $\varphi \in T$. The set $\mathrm{Th}\mathcal{L}$ of all $\mathcal{L}$-theories forms a complete lattice under set-inclusion.

Logics

Let **Fm** be the formula algebra in some language over a countably infinite set Var of variables.

Definition

A logic is a pair $\mathcal{L} = \langle \mathbf{Fm}, \vdash_{\mathcal{L}} \rangle$, where **Fm** is the formula algebra and $\vdash_{\mathcal{L}}$ is a relation between sets of formulas and single formulas satisfying the following:

(i) if $\varphi \in \Gamma$, then $\Gamma \vdash_{\mathcal{L}} \varphi$; (reflexivity)

(ii) if $\Gamma \vdash_{\mathcal{L}} \varphi$ and $\Delta \vdash_{\mathcal{L}} \gamma$ for all $\gamma \in \Gamma$, then $\Delta \vdash_{\mathcal{L}} \varphi$; (cut)

(iii) if $\Gamma \vdash_{\mathcal{L}} \varphi$ and $\sigma \in \mathrm{End}(\mathbf{Fm})$, then $\sigma\Gamma \vdash_{\mathcal{L}} \sigma\varphi$. (structurality)

A logic is finitary if moreover the following holds

(iv) if $\Gamma \vdash_{\mathcal{L}} \varphi$, then there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathcal{L}} \varphi$.

A set T of formulas is an $\mathcal{L}$-theory if it is closed under the consequence relation of $\mathcal{L}$, i.e. if $T \vdash_{\mathcal{L}} \varphi$ implies $\varphi \in T$. The set $\mathrm{Th}\mathcal{L}$ of all $\mathcal{L}$-theories forms a complete lattice under set-inclusion.

Logics

Let **Fm** be the formula algebra in some language over a countably infinite set Var of variables.

Definition

A logic is a pair $\mathcal{L} = \langle \mathbf{Fm}, \vdash_{\mathcal{L}} \rangle$, where **Fm** is the formula algebra and $\vdash_{\mathcal{L}}$ is a relation between sets of formulas and single formulas satisfying the following:

(i) if $\varphi \in \Gamma$, then $\Gamma \vdash_{\mathcal{L}} \varphi$; (reflexivity)

(ii) if $\Gamma \vdash_{\mathcal{L}} \varphi$ and $\Delta \vdash_{\mathcal{L}} \gamma$ for all $\gamma \in \Gamma$, then $\Delta \vdash_{\mathcal{L}} \varphi$; (cut)

(iii) if $\Gamma \vdash_{\mathcal{L}} \varphi$ and $\sigma \in \mathrm{End}(\mathbf{Fm})$, then $\sigma\Gamma \vdash_{\mathcal{L}} \sigma\varphi$. (structurality)

A logic is finitary if moreover the following holds

(iv) if $\Gamma \vdash_{\mathcal{L}} \varphi$, then there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathcal{L}} \varphi$.

A set T of formulas is an $\mathcal{L}$-theory if it is closed under the consequence relation of $\mathcal{L}$, i.e. if $T \vdash_{\mathcal{L}} \varphi$ implies $\varphi \in T$. The set $\mathrm{Th}\mathcal{L}$ of all $\mathcal{L}$-theories forms a complete lattice under set-inclusion.

Leibniz operator

One central aim in AAL is to classify logics according to the properties that the so called Leibniz operator has when restricted to (a sublattice of) the lattice of theories of a given logic.

Definition

Let $\Gamma \subseteq \mathrm{Fm}$. The Leibniz congruence $\Omega\Gamma$ determined by Γ is defined as

$\langle \alpha, \beta \rangle \in \Omega\Gamma$ if for all formulas φ and all variables x,

$$\varphi(x/\alpha) \in \Gamma \text{ if and only if } \varphi(x/\beta) \in \Gamma.$$

The Leibniz operator Ω is the mapping that assigns to any subset Γ of formulas the Leibniz congruence $\Omega\Gamma$.

Leibniz operator

One central aim in AAL is to classify logics according to the properties that the so called Leibniz operator has when restricted to (a sublattice of) the lattice of theories of a given logic.

Definition

Let $\Gamma \subseteq \mathrm{Fm}$. The Leibniz congruence $\Omega\Gamma$ determined by Γ is defined as

$\langle \alpha, \beta \rangle \in \Omega\Gamma$ if for all formulas φ and all variables x,

$$\varphi(x/\alpha) \in \Gamma \text{ if and only if } \varphi(x/\beta) \in \Gamma.$$

The Leibniz operator Ω is the mapping that assigns to any subset Γ of formulas the Leibniz congruence $\Omega\Gamma$.

Leibniz operator

One central aim in AAL is to classify logics according to the properties that the so called Leibniz operator has when restricted to (a sublattice of) the lattice of theories of a given logic.

Definition

Let $\Gamma \subseteq \mathrm{Fm}$. The Leibniz congruence $\Omega\Gamma$ determined by Γ is defined as

$$\langle \alpha, \beta \rangle \in \Omega\Gamma \text{ if for all formulas } \varphi \text{ and all variables } x, \quad \varphi(x/\alpha) \in \Gamma \text{ if and only if } \varphi(x/\beta) \in \Gamma.$$

The Leibniz operator Ω is the mapping that assigns to any subset Γ of formulas the Leibniz congruence $\Omega\Gamma$.

Protoalgebraic Logics

Definition

A logic $\mathcal{L}$ is protoalgebraic if for every $\mathcal{L}$-theory T and all formulas φ and ψ,

$$\langle \varphi, \psi \rangle \in \Omega T \text{ implies } \varphi, T \dashv\vdash_{\mathcal{L}} T, \psi.$$

Theorem

Let $\mathcal{L}$ be a logic. Then the following are equivalent:

(i) *$\mathcal{L}$ is protoalgebraic;*

(ii) *Ω is monotone on $\mathrm{Th}\mathcal{L}$.*

Protoalgebraic Logics

Definition

A logic $\mathcal{L}$ is protoalgebraic if for every $\mathcal{L}$-theory T and all formulas φ and ψ,

$$\langle \varphi, \psi \rangle \in \Omega T \text{ implies } \varphi, T \dashv\vdash_{\mathcal{L}} T, \psi.$$

Theorem

Let $\mathcal{L}$ be a logic. Then the following are equivalent:

(i) *$\mathcal{L}$ is protoalgebraic;*

(ii) *Ω is monotone on $\mathrm{Th}\mathcal{L}$.*

Two Syntactic Characterizations

There are two distinct syntactic characterizations for protoalgebraic logics via the existence a set of formulas with certain properties. We will use one of them to define finitely protoalgebraic logics.

Definition

Let $\mathcal{L}$ be a logic. A set $\Delta(x, y)$ of formulas in two variables is a proto-implication for $\mathcal{L}$ if the following two conditions hold:

(i) $\vdash_{\mathcal{L}} \Delta(x, x)$;

(ii) $x, \Delta(x, y) \vdash_{\mathcal{L}} y$.

Two Syntactic Characterizations

There are two distinct syntactic characterizations for protoalgebraic logics via the existence a set of formulas with certain properties. We will use one of them to define finitely protoalgebraic logics.

Definition

Let $\mathcal{L}$ be a logic. A set $\Delta(x, y)$ of formulas in two variables is a proto-implication for $\mathcal{L}$ if the following two conditions hold:

(i) $\vdash_{\mathcal{L}} \Delta(x, x)$;

(ii) $x, \Delta(x, y) \vdash_{\mathcal{L}} y$.

Two Syntactic Characterizations

Given a set $\Delta(x, y, \bar{z})$ of formulas with main variables x and y and parameters $\bar{z}$ we define for all formulas φ and ψ,

$$\Delta(\langle\varphi, \psi\rangle) := \{\delta(\varphi, \psi, \bar{\gamma})\colon \delta(x, y, \bar{z}) \in \Delta(x, y, \bar{z}), \bar{\gamma} \in \mathrm{Fm}\}.$$

Definition

Let $\mathcal{L}$ be a logic. A set $\Delta(x, y, \bar{z})$ is a parameterized equivalence for $\mathcal{L}$ if the following three conditions hold:

(i) $\vdash_{\mathcal{L}} \Delta(\langle x, x\rangle)$;

(ii) $x, \Delta(\langle x, y\rangle) \vdash_{\mathcal{L}} y$;

(iii) $\Delta(\langle x_1, y_1\rangle), \ldots, \Delta(\langle x_n, y_n\rangle) \vdash_{\mathcal{L}} \Delta(\langle \lambda x_1 \ldots x_n, \lambda y_1 \ldots y_n\rangle)$ for all n-ary connectives λ.

Two Syntactic Characterizations

Given a set $\Delta(x, y, \bar{z})$ of formulas with main variables x and y and parameters $\bar{z}$ we define for all formulas φ and ψ,

$$\Delta(\langle \varphi, \psi \rangle) := \{\delta(\varphi, \psi, \bar{\gamma})\colon \delta(x, y, \bar{z}) \in \Delta(x, y, \bar{z}), \bar{\gamma} \in \mathrm{Fm}\}.$$

Definition

Let $\mathcal{L}$ be a logic. A set $\Delta(x, y, \bar{z})$ is a parameterized equivalence for $\mathcal{L}$ if the following three conditions hold:

(i) $\vdash_{\mathcal{L}} \Delta(\langle x, x \rangle)$;

(ii) $x, \Delta(\langle x, y \rangle) \vdash_{\mathcal{L}} y$;

(iii) $\Delta(\langle x_1, y_1 \rangle), \ldots, \Delta(\langle x_n, y_n \rangle) \vdash_{\mathcal{L}} \Delta(\langle \lambda x_1 \ldots x_n, \lambda y_1 \ldots y_n \rangle)$ for all n-ary connectives λ.

Two Syntactic Characterizations

Theorem

Let $\mathcal{L}$ be a logic. Then the following are equivalent:

(i) *$\mathcal{L}$ is protoalgebraic;*

(ii) *There is a proto-implication for $\mathcal{L}$;*

(iii) *There is a parameterized equivalence for $\mathcal{L}$.*

Definition

A logic $\mathcal{L}$ is (finitely) equivalential if there is a (finite) parameter-free equivalence for $\mathcal{L}$.

Two Syntactic Characterizations

Theorem

Let $\mathcal{L}$ be a logic. Then the following are equivalent:

(i) *$\mathcal{L}$ is protoalgebraic;*

(ii) *There is a proto-implication for $\mathcal{L}$;*

(iii) *There is a parameterized equivalence for $\mathcal{L}$.*

Definition

A logic $\mathcal{L}$ is (finitely) equivalential if there is a (finite) parameter-free equivalence for $\mathcal{L}$.

Two Syntactic Characterizations

Which of the two syntactic characterizations do we use to define finitely protoalgebraic logics? The following lemma suggests that the second one is to be preferred.

Lemma

Let $\mathcal{L}$ be a protoalgebraic logic and let $\Delta(x, y, \bar{z})$ be a parameterized equivalence for $\mathcal{L}$. Then for any $T \in \mathrm{Th}\mathcal{L}$,

$$\langle \varphi, \psi \rangle \in \Omega T \text{ if and only if } \Delta(\langle \varphi, \psi \rangle) \subseteq T.$$

Hence, we call a logic finitely protoalgebraic if it has a finite parameterized equivalence.

Lemma

Let $\mathcal{L}$ be a logic. $\mathcal{L}$ has a finite proto-implication if and only if the finitary companion $\mathcal{L}^f$ is protoalgebraic.

Two Syntactic Characterizations

Which of the two syntactic characterizations do we use to define finitely protoalgebraic logics? The following lemma suggests that the second one is to be preferred.

Lemma

Let $\mathcal{L}$ be a protoalgebraic logic and let $\Delta(x, y, \bar{z})$ be a parameterized equivalence for $\mathcal{L}$. Then for any $T \in \mathrm{Th}\mathcal{L}$,

$$\langle \varphi, \psi \rangle \in \Omega T \text{ if and only if } \Delta(\langle \varphi, \psi \rangle) \subseteq T.$$

Hence, we call a logic finitely protoalgebraic if it has a finite parameterized equivalence.

Lemma

Let $\mathcal{L}$ be a logic. $\mathcal{L}$ has a finite proto-implication if and only if the finitary companion $\mathcal{L}^f$ is protoalgebraic.

Two Syntactic Characterizations

Which of the two syntactic characterizations do we use to define finitely protoalgebraic logics? The following lemma suggests that the second one is to be preferred.

Lemma

Let $\mathcal{L}$ be a protoalgebraic logic and let $\Delta(x, y, \bar{z})$ be a parameterized equivalence for $\mathcal{L}$. Then for any $T \in \mathrm{Th}\mathcal{L}$,

$$\langle \varphi, \psi \rangle \in \Omega T \text{ if and only if } \Delta(\langle \varphi, \psi \rangle) \subseteq T.$$

Hence, we call a logic finitely protoalgebraic if it has a finite parameterized equivalence.

Lemma

Let $\mathcal{L}$ be a logic. $\mathcal{L}$ has a finite proto-implication if and only if the finitary companion $\mathcal{L}^f$ is protoalgebraic.

Two Syntactic Characterizations

Which of the two syntactic characterizations do we use to define finitely protoalgebraic logics? The following lemma suggests that the second one is to be preferred.

Lemma

Let $\mathcal{L}$ be a protoalgebraic logic and let $\Delta(x, y, \bar{z})$ be a parameterized equivalence for $\mathcal{L}$. Then for any $T \in \mathrm{Th}\mathcal{L}$,

$$\langle \varphi, \psi \rangle \in \Omega T \textit{ if and only if } \Delta(\langle \varphi, \psi \rangle) \subseteq T.$$

Hence, we call a logic finitely protoalgebraic if it has a finite parameterized equivalence.

Lemma

Let $\mathcal{L}$ be a logic. $\mathcal{L}$ has a finite proto-implication if and only if the finitary companion $\mathcal{L}^f$ is protoalgebraic.

Counterexample: The Doubting Thomas Logic

We define a logic that is finitely protoalgebraic and equivalential, but not finitely equivalential. Consider a language with single ternary connective λ and define

$$\Delta(x, y) := \{\lambda(x, y, \delta) \colon \delta \in \mathrm{Fm}(x, y)\}.$$

The Doubting Thomas Logic $\mathcal{DT}$ is the least logic satisfying the following:

(i) $\vdash \Delta(x, x)$;

(ii) $x, \Delta(x, y) \vdash y$;

(iii) $\Delta(x_1, y_1), \Delta(x_2, y_2), \Delta(x_3, y_3) \vdash \lambda(\lambda(x_1, x_2, x_3), \lambda(y_1, y_2, y_3), z)$.

Now $\{\lambda(x, y, z)\}$ is a parameterized equivalence and $\Delta(x, y)$ is a parameter-free equivalence for $\mathcal{DT}$. On the other hand $\mathcal{DT}$ does not have a finite parameter-free equivalence, since for no finite $\Delta'(x, y) \subseteq \Delta(x, y)$ does the Modus Ponens hold. Also, $\mathcal{DT}$ does not have a finite proto-implication.

Counterexample: The Doubting Thomas Logic

We define a logic that is finitely protoalgebraic and equivalential, but not finitely equivalential. Consider a language with single ternary connective λ and define

$$\Delta(x, y) := \{\lambda(x, y, \delta) \colon \delta \in \mathrm{Fm}(x, y)\}.$$

The Doubting Thomas Logic $\mathcal{DT}$ is the least logic satisfying the following:

(i) $\vdash \Delta(x, x)$;

(ii) $x, \Delta(x, y) \vdash y$;

(iii) $\Delta(x_1, y_1), \Delta(x_2, y_2), \Delta(x_3, y_3) \vdash \lambda(\lambda(x_1, x_2, x_3), \lambda(y_1, y_2, y_3), z)$.

Now $\{\lambda(x, y, z)\}$ is a parameterized equivalence and $\Delta(x, y)$ is a parameter-free equivalence for $\mathcal{DT}$. On the other hand $\mathcal{DT}$ does not have a finite parameter-free equivalence, since for no finite $\Delta'(x, y) \subseteq \Delta(x, y)$ does the Modus Ponens hold. Also, $\mathcal{DT}$ does not have a finite proto-implication.

Counterexample: The Doubting Thomas Logic

We define a logic that is finitely protoalgebraic and equivalential, but not finitely equivalential. Consider a language with single ternary connective λ and define

$$\Delta(x, y) := \{\lambda(x, y, \delta) \colon \delta \in \mathrm{Fm}(x, y)\}.$$

The Doubting Thomas Logic $\mathcal{DT}$ is the least logic satisfying the following:

(i) $\vdash \Delta(x, x)$;

(ii) $x, \Delta(x, y) \vdash y$;

(iii) $\Delta(x_1, y_1), \Delta(x_2, y_2), \Delta(x_3, y_3) \vdash \lambda(\lambda(x_1, x_2, x_3), \lambda(y_1, y_2, y_3), z)$.

Now $\{\lambda(x, y, z)\}$ is a parameterized equivalence and $\Delta(x, y)$ is a parameter-free equivalence for $\mathcal{DT}$. On the other hand $\mathcal{DT}$ does not have a finite parameter-free equivalence, since for no finite $\Delta'(x, y) \subseteq \Delta(x, y)$ does the Modus Ponens hold. Also, $\mathcal{DT}$ does not have a finite proto-implication.

Counterexample: The Doubting Thomas Logic

We define a logic that is finitely protoalgebraic and equivalential, but not finitely equivalential. Consider a language with single ternary connective λ and define

$$\Delta(x, y) := \{\lambda(x, y, \delta) \colon \delta \in \mathrm{Fm}(x, y)\}.$$

The Doubting Thomas Logic $\mathcal{DT}$ is the least logic satisfying the following:

(i) $\vdash \Delta(x, x)$;

(ii) $x, \Delta(x, y) \vdash y$;

(iii) $\Delta(x_1, y_1), \Delta(x_2, y_2), \Delta(x_3, y_3) \vdash \lambda(\lambda(x_1, x_2, x_3), \lambda(y_1, y_2, y_3), z)$.

Now $\{\lambda(x, y, z)\}$ is a parameterized equivalence and $\Delta(x, y)$ is a parameter-free equivalence for $\mathcal{DT}$. On the other hand $\mathcal{DT}$ does not have a finite parameter-free equivalence, since for no finite $\Delta'(x, y) \subseteq \Delta(x, y)$ does the Modus Ponens hold. Also, $\mathcal{DT}$ does not have a finite proto-implication.

Invariant theories

Definition

Let $\mathcal{L}$ be a logic and let X be a set of variables. An $\mathcal{L}$-theory T is X-invariant if $\sigma T \subseteq T$ for any substitution σ such that $\sigma x = x$ for all $x \in X$.

We denote the set of all X-invariant $\mathcal{L}$-theories by $\mathrm{Th}^{X}_{\mathrm{inv}}\mathcal{L}$. $\mathrm{Th}^{X}_{\mathrm{inv}}\mathcal{L}$ is a complete sublattice of $\mathrm{Th}\mathcal{L}$ for any set X of variables.
In the following we are interested in the lattice $\mathrm{Th}^{xy}_{\mathrm{inv}}\mathcal{L}$ of all $\{x, y\}$-invariant $\mathcal{L}$-theories.

Invariant theories

Definition

Let $\mathcal{L}$ be a logic and let X be a set of variables. An $\mathcal{L}$-theory T is X-invariant if $\sigma T \subseteq T$ for any substitution σ such that $\sigma x = x$ for all $x \in X$.

We denote the set of all X-invariant $\mathcal{L}$-theories by $\mathrm{Th}^X_{\mathrm{inv}}\mathcal{L}$. $\mathrm{Th}^X_{\mathrm{inv}}\mathcal{L}$ is a complete sublattice of $\mathrm{Th}\mathcal{L}$ for any set X of variables.

In the following we are interested in the lattice $\mathrm{Th}^{xy}_{\mathrm{inv}}\mathcal{L}$ of all $\{x, y\}$-invariant $\mathcal{L}$-theories.

Invariant theories

Definition

Let $\mathcal{L}$ be a logic and let X be a set of variables. An $\mathcal{L}$-theory T is X-invariant if $\sigma T \subseteq T$ for any substitution σ such that $\sigma x = x$ for all $x \in X$.

We denote the set of all X-invariant $\mathcal{L}$-theories by $\mathrm{Th}^{X}_{\mathrm{inv}}\mathcal{L}$. $\mathrm{Th}^{X}_{\mathrm{inv}}\mathcal{L}$ is a complete sublattice of $\mathrm{Th}\mathcal{L}$ for any set X of variables.
In the following we are interested in the lattice $\mathrm{Th}^{xy}_{\mathrm{inv}}\mathcal{L}$ of all $\{x, y\}$-invariant $\mathcal{L}$-theories.

Finitely protoalgebraic logics

Lemma

Let $\mathcal{L}$ be a logic. Then the following are equivalent:

(i) Ω *is monotone on* $\mathrm{Th}\mathcal{L}$;

(ii) Ω *is monotone on* $\mathrm{Th}^{xy}_{\mathrm{inv}}\mathcal{L}$.

Theorem

Let $\mathcal{L}$ be a logic. Then the following are equivalent:

(i) $\mathcal{L}$ *is finitely protoalgebraic;*

(ii) Ω *is continuous on* $\mathrm{Th}^{xy}_{\mathrm{inv}}\mathcal{L}$, *i.e. for any directed family* $\{T_i : i \in I\}$ *of* $\{x, y\}$*-invariant* $\mathcal{L}$*-theories such that* $\bigcup_{i \in I} T_i$ *is an* $\mathcal{L}$*-theory, it holds that*

$$\Omega \bigcup_{i \in I} T_i = \bigcup_{i \in I} \Omega T_i.$$

Finitely protoalgebraic logics

Lemma

Let $\mathcal{L}$ be a logic. Then the following are equivalent:

(i) *Ω is monotone on $\mathrm{Th}\mathcal{L}$;*

(ii) *Ω is monotone on $\mathrm{Th}^{xy}_{\mathrm{inv}}\mathcal{L}$.*

Theorem

Let $\mathcal{L}$ be a logic. Then the following are equivalent:

(i) *$\mathcal{L}$ is finitely protoalgebraic;*

(ii) *Ω is continuous on $\mathrm{Th}^{xy}_{\mathrm{inv}}\mathcal{L}$, i.e. for any directed family $\{T_i : i \in I\}$ of $\{x, y\}$-invariant $\mathcal{L}$-theories such that $\bigcup_{i \in I} T_i$ is an $\mathcal{L}$-theory, it holds that*

$$\Omega \bigcup_{i \in I} T_i = \bigcup_{i \in I} \Omega T_i.$$

Finitely protoalgebraic logics

Lemma

Let $\mathcal{L}$ be a logic. Then the following are equivalent:

(i) Ω *is monotone on* $\mathrm{Th}\mathcal{L}$;

(ii) Ω *is monotone on* $\mathrm{Th}^{xy}_{\mathrm{inv}}\mathcal{L}$.

Theorem

Let $\mathcal{L}$ be a logic. Then the following are equivalent:

(i) $\mathcal{L}$ *is finitely protoalgebraic;*

(ii) Ω *is continuous on* $\mathrm{Th}^{xy}_{\mathrm{inv}}\mathcal{L}$, *i.e. for any directed family* $\{T_i : i \in I\}$ *of* $\{x, y\}$*-invariant* $\mathcal{L}$*-theories such that* $\bigcup_{i\in I} T_i$ *is an* $\mathcal{L}$*-theory, it holds that*

$$\Omega \bigcup_{i\in I} T_i = \bigcup_{i\in I} \Omega T_i.$$

Finitely weakly algebraizable logics

Definition

A (finitely) protoalgebraic logic is (finitely) weakly algebraizable if Ω is injective on $\mathrm{Th}\mathcal{L}$.

Lemma

Let $\mathcal{L}$ be a protoalgebraic logic. Then the following are equivalent:

(i) *Ω is injective on $\mathrm{Th}\mathcal{L}$;*

(ii) *Ω is injective on $\mathrm{Th}^{xy}_{\mathrm{inv}}\mathcal{L}$.*

Theorem

Let $\mathcal{L}$ be a logic. Then the following are equivalent:

(i) *$\mathcal{L}$ is finitely weakly algebraizable;*

(ii) *Ω is continuous and injective on $\mathrm{Th}^{xy}_{\mathrm{inv}}\mathcal{L}$.*

Finitely weakly algebraizable logics

Definition

A (finitely) protoalgebraic logic is (finitely) weakly algebraizable if Ω is injective on $\mathrm{Th}\mathcal{L}$.

Lemma

Let $\mathcal{L}$ be a protoalgebraic logic. Then the following are equivalent:

(i) *Ω is injective on $\mathrm{Th}\mathcal{L}$;*

(ii) *Ω is injective on $\mathrm{Th}^{xy}_{\mathrm{inv}}\mathcal{L}$.*

Theorem

Let $\mathcal{L}$ be a logic. Then the following are equivalent:

(i) *$\mathcal{L}$ is finitely weakly algebraizable;*

(ii) *Ω is continuous and injective on $\mathrm{Th}^{xy}_{\mathrm{inv}}\mathcal{L}$.*

Finitely weakly algebraizable logics

Definition

A (finitely) protoalgebraic logic is (finitely) weakly algebraizable if Ω is injective on $\mathrm{Th}\mathcal{L}$.

Lemma

Let $\mathcal{L}$ be a protoalgebraic logic. Then the following are equivalent:

(i) *Ω is injective on $\mathrm{Th}\mathcal{L}$;*

(ii) *Ω is injective on $\mathrm{Th}^{xy}_{\mathrm{inv}}\mathcal{L}$.*

Theorem

Let $\mathcal{L}$ be a logic. Then the following are equivalent:

(i) *$\mathcal{L}$ is finitely weakly algebraizable;*

(ii) *Ω is continuous and injective on $\mathrm{Th}^{xy}_{\mathrm{inv}}\mathcal{L}$.*

The End

Thank you!