Poset Product and BL-chains

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Hoops and BL-algebras

A hoop is an algebra $\mathbf{H} = \langle H, \cdot, \to, 1\rangle$ of type $\langle 2, 2, 0\rangle$ such that $\langle H, \cdot, 1\rangle$ is a commutative monoid satisfying

(i) $x \to x = 1$

(ii) $x \cdot (x \to y) = y \cdot (y \to x)$

(iii) $x \to (y \to z) = (x \cdot y) \to z$

for all $x, y, z \in H$.

If $\mathbf{H}$ is a hoop, then $(H, \cdot, 1)$ is a naturally ordered residuated commutative monoid, where $x \leq y$ if and only if $x \to y = 1$ and the residuation is

$$x \cdot y \leq z \text{ if and only if } x \leq y \to z.$$

Hoops and BL-algebras

A hoop is called

- bounded if it is an algebra $\mathbf{H} = \langle H, \cdot, \rightarrow, 0, 1 \rangle$ such that $\langle H, \cdot, \rightarrow, 1 \rangle$ is a hoop and $0 \leq x$ for all $x \in H$.
- basic if it is a hoop satisfying the identity

$$(((x \rightarrow y) \rightarrow z) \cdot ((y \rightarrow x) \rightarrow z)) \rightarrow z = 1.$$

- a Wajsberg hoop if it satisfies

$$(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x.$$

The prelineariry equation $(x \rightarrow y) \vee (y \rightarrow x) = 1$ holds in every basic hoop.

Hoops and BL-algebras

A BL-algebra is a bounded basic hoop and a BL-chain is a totally ordered BL-algebra. We will mainly work with two subvarieties of BL-algebras

- the subvariety of *MV-algebras*, characterized by

$$\neg\neg x = x \qquad (\text{where } \neg x = x \to 0).$$

- the subvariety of *product algebras*, characterized by

$$(\neg\neg z \cdot ((x \cdot z) \to (y \cdot z))) \to (x \to y) = 1$$
$$x \wedge \neg x = 0$$

An MV-chain is a totally ordered MV-algebra and a product chain is a totally ordered product algebra.

Classical examples

The *standard MV-chain* $\mathbf{[0,1]}_{\mathrm{MV}}$ is the MV-algebra whose universe is the real unit interval $[0,1]$, where $x \cdot y = \max(0, x+y-1)$ and $x \to y = \min(1, 1-x+y)$. For $n \geq 2$, $\mathbf{Ł}_n$ is the subalgebra of $\mathbf{[0,1]}_{\mathrm{MV}}$ with domain

$$Ł_n = \left\{ \frac{0}{n-1}, \frac{1}{n-1}, \frac{2}{n-1}, \ldots, \frac{n-1}{n-1} \right\}.$$

The *standard product chain* is the algebra $\mathbf{[0,1]}_{\Pi} = \langle [0,1], \cdot, \to, 0, 1 \rangle$ where $\cdot$ is the usual product over the real interval $[0,1]$ and $\to$ is given by

$$x \to y = \begin{cases} y/x & \text{if } x > y; \\ 1 & \text{if } x \leq y. \end{cases}$$

Ordinal sum

Let $\{\mathbf{H}_i : i \in I\}$ be a family of hoops indexed by a totally ordered set $(I, \leq)$. Let us assume that $\mathbf{H}_i \cap \mathbf{H}_j = \{1\}$ whenever $i \neq j \in I$. The ordinal sum of this family is the hoop

$$\bigoplus_{i \in I} \mathbf{H}_i = \langle \bigcup_{i \in I} H_i, \cdot, \rightarrow, 1 \rangle,$$

where the operations are given by

$$x \cdot y = \begin{cases} x \cdot_i y & \text{if } x, y \in H_i, \\ x & \text{if } x \in H_i \setminus \{1\}, y \in H_j, i < j, \\ y & \text{if } y \in H_i \setminus \{1\}, x \in H_j, i < j. \end{cases}$$

$$x \rightarrow y = \begin{cases} 1 & \text{if } x \in H_i \setminus \{1\}, y \in W_j, i < j, \\ x \rightarrow_i y & \text{if } x, y \in H_i, \\ y & \text{if } y \in H_i, x \in H_j, i < j. \end{cases}$$

BL-chain decomposition

Decomposition theorem for BL-chains (Aglianò-Montagna)

Each non-trivial BL-chain admits a unique decomposition into an ordinal sum of non-trivial totally ordered Wajsberg hoops.

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Remarks

If $\bigoplus_{i \in I} \mathbf{W}_i$ is the decomposition of a BL-chain into Wajsberg hoops, then the index set I has a minimum element i_0 and the resulting constant bottom in the ordinal sum is the bottom of $\mathbf{W}_{i_0}$.

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Remarks

Totally ordered Wajsberg hoops can be either lower bounded or not.

- If bounded, they are bottom free reducts of MV-chains.
- If unbounded, they are *cancellative* Wajsberg hoops, i.e. they satisfy the identity $x \to (x \cdot y) = y$. Example: $\mathbf{(0,1]_\Pi}$.

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Remarks

$[0,1]_{\Pi} \cong \text{Ł}_2 \oplus (0,1]_{\Pi}$. In general, if $\mathbf{A}$ is a product chain, then

$$\mathbf{A} \cong \text{Ł}_2 \oplus \mathbf{W},$$

where $\mathbf{W}$ is a cancellative hoop. In addition, for each cancellative totally ordered hoop $\mathbf{W}$, the ordinal sum $\text{Ł}_2 \oplus \mathbf{W}$ is a product chain.

Poset product

Given a poset $\mathbf{P} = \langle P, \leq \rangle$ and a collection $\{\mathbf{A}_p : p \in P\}$ of BL-algebras sharing the same neutral element 1 and the same minimum element 0, the poset product $\bigotimes_{p \in P} \mathbf{A}_p$ is the residuated lattice $\mathbf{A} = \langle A, \cdot, \rightarrow, \vee, \wedge, \bot, \top \rangle$ defined as follows:

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- The domain of $\mathbf{A}$ is the set of all maps $x \in \prod_{p \in P} A_p$ such that for all $i \in P$, if $x_i \neq 1$, then $x_j = 0$ provided that $j > i$.

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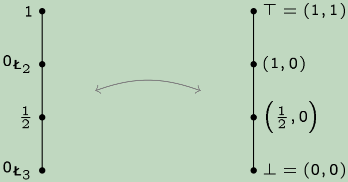
- $\top$ is the map whose value in each coordinate is 1. Analogously for the symbol $\bot$ to denote the minimum element.
- Monoid and lattice operations are defined pointwise.
- The residual is

$$(x \rightarrow_{\mathbf{A}} y)_i = \begin{cases} x_i \rightarrow_{\mathbf{A}_i} y_i & \text{if } x_j \leq y_j \text{ for all } j < i; \\ 0 & \text{otherwise.} \end{cases}$$

Properties and examples

If P is finite and totally ordered, then $\bigotimes_{i \in P} \mathbf{A}_i \cong \bigoplus_{i \in P} \mathbf{A}_i$.

Let $P = \{a < b\}$, $\mathbf{A}_a = Ł_3$ and $\mathbf{A}_b = Ł_2$, then $Ł_3 \otimes Ł_2 \cong Ł_3 \oplus Ł_2$.



$\left(\frac{1}{2}, 0\right) \cdot \left(\frac{1}{2}, 0\right) = \bot$

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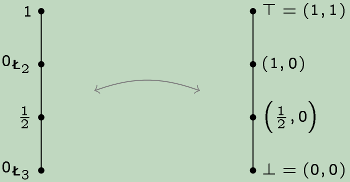

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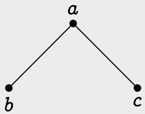
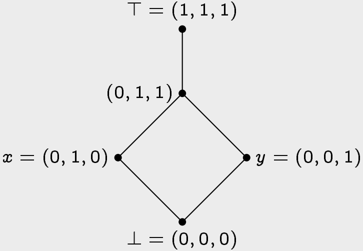
If P is an antichain, then $\bigotimes_{i \in P} \mathbf{A}_i = \prod_{i \in P} \mathbf{A}_i$.

Let $P = \{a \parallel b\}$ and $\mathbf{A}_a = \mathbf{A}_b = \text{Ł}_2$, then $\text{Ł}_2 \otimes \text{Ł}_2 = \text{Ł}_2 \times \text{Ł}_2$.

Properties and examples

If $\mathbf{\Lambda} = \langle \Lambda, < \rangle = \langle \{a, b, c\}, \{(b, a), (c, a)\} \rangle$ and $\mathbf{A}_a = \mathbf{A}_b = \mathbf{A}_c = \textbf{\L}_2$, then





The poset product of the family is

$$\bigotimes_{\Lambda} \textbf{\L}_2 = \{(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,1,1)\}.$$

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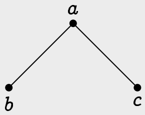

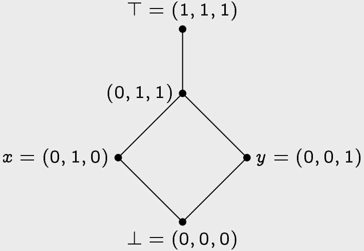

$\bigotimes_{\mathbf{\Lambda}} \text{\L}_2$ is not a BL-algebra because

$$(x \to y) \vee (y \to x) = (0,0,1) \vee (0,1,0) = (0,1,1) < (1,1,1) = \top.$$

Forests

From now on, we will consider posets that do not contain as a subposet the configuration Λ. They are known as forests. Thus, a forest is a poset $\mathbf{P} = \langle P, \leq \rangle$ such that for each $i \in P$, the downset

$$\downarrow i = \{j \in P : j \leq i\}$$

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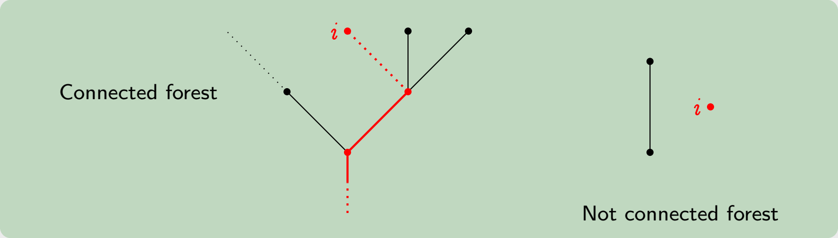

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Theorem

If P is a forest and $\mathbf{A}_p$ is a BL-chain for all $p \in P$, then $\bigotimes_{p \in P} \mathbf{A}_p$ is a BL-algebra.

Idempotent free BL-algebras

An algebra $\mathbf{A}$ is said to be poset product indecomposable if $\mathbf{A}$ is non-trivial and if $\mathbf{A}$ is a poset product of two algebras $\mathbf{A}_1$ and $\mathbf{A}_2$, then either $\mathbf{A}_1$ or $\mathbf{A}_2$ is trivial.

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Proposition

Let $\mathbf{A}$ be a non-trivial BL-chain. Then

$$\mathbf{A} \text{ is idempotent free} \iff \mathbf{A} \text{ is poset product indecomposable.}$$

For all $n \geq 2$, $\text{Ł}_n \oplus (0, 1]_{\Pi}$ is indecomposable in the sense of poset product.

Representability

Given a BL-chain $\mathbf{A}$, if there are a totally ordered set P and a family of idempotent free BL-chains $\{\mathbf{A}_i : i \in P\}$ such that $\mathbf{A} \cong \bigotimes_{i \in P} \mathbf{A}_i$, we will say that $\mathbf{A}$ is representable. If the family only contains MV-chains and product chains, we will say that $\mathbf{A}$ is ΠMV-representable.

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Jipsen-Montagna's generalization for Di Nola-Lettieri's result

Every finite BL-algebra is isomorphic to the a poset product of a collection of MV-chains.

Poset product of idempotent free BL-chains

Theorem

Let $\langle P, \leq \rangle$ be a totally ordered set and $\{\mathbf{A}_p : p \in P\}$ be a family of idempotent free BL-chains. Then $\bigoplus_{p\in P} \mathbf{A}_p \cong \bigotimes_{p\in P} \mathbf{A}_p$ if and only if P is well-ordered.

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$(\Rightarrow)$ If $\bigoplus_{p \in P} \mathbf{A}_p \cong \bigotimes_{p \in P} \mathbf{A}_p$, since $\mathrm{Id}(\mathbf{A}_p) = \{0, 1\}\ \forall p \in P$,

$$\bigoplus_{P} \mathbf{Ł}_2 \cong \bigotimes_{P} \mathbf{Ł}_2.$$

Given that $\bigotimes_P \mathbf{Ł}_2$ is complete, P can be seen as a complete poset which actually is a well-ordered set.

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($\Leftarrow$) If P is a well-ordered set, the map $f \colon \bigoplus_{p \in P} \mathbf{A}_p \to \bigotimes_{p \in P} \mathbf{A}_p$ defined by $f(1) = \top$ and

$$f(a)_p = \begin{cases} 1 & \text{if } p < j; \\ a & \text{if } p = j; \\ 0 & \text{if } p > j. \end{cases}$$

if $a \in A_j \setminus \{\top\}$ is an isomorphism.

Some issues

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Representable BL-chain without a well-ordered index set

Let $\mathbf{A} = \bigoplus_I \text{Ł}_2$, where $\mathbf{I} = \langle \{b\} \cup \mathbb{Z}^-, \leq \rangle$. Although I is not a well-ordered set, $\mathbf{A} \cong \bigotimes_{\mathbb{Z}^-} \text{Ł}_2$. Observe that $\bigoplus_{\mathbb{Z}^-} \text{Ł}_2 \ncong \bigotimes_{\mathbb{Z}^-} \text{Ł}_2$.

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Non-representable BL-chain indexed by a well-ordered set

Let $\mathbf{A} = \bigoplus_{i \in I} \mathbf{W}_i$, where $\mathbf{I} = \langle \mathbb{N} \cup \{t\}, \leq \rangle$, $\mathbf{W}_n = \text{Ł}_2$ for all $n \in \mathbb{N}$ and $\mathbf{W}_t = (\mathbf{0}, \mathbf{1}]_{\mathbf{\Pi}}$. Then $\mathbf{A}$ is not representable. Note that $\mathbf{W}_t$ is not a BL-chain.

A sufficient (but strong) condition for representability

Proposition

If each prime filter in a BL-chain $\mathbf{A}$ is a principal filter, then $\mathbf{A}$ is representable.

If $\mathbf{A} \cong \bigoplus_{i \in I} \mathbf{W}_i$, it turns out that the index set I is well-ordered and every $\mathbf{W}_i$ is a bounded hoop (MV-chain). Thus $\mathbf{A} \cong \bigotimes_{i \in I} \mathbf{W}_i$.

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Since in a finite BL-algebra all filters are principal, this is a proposition that (for the case of BL-chains) enhances the Jipsen and Montagna's result we cited before. However, it must be said that the hypothesis is still too restrictive, since in general idempotent free BL-chains contain a non-prime principal filter.

For all $n \geq 2$, the set $(0, 1]$ is a prime filter in the representable BL-chain $\mathbf{Ł}_n \oplus \mathbf{(0,1]_\Pi}$ which is not a principal filter.

Saturated BL-chains

Let $\mathbf{A}$ be a BL-chain. A pair of sets (X, Y) is called a cut in $\mathbf{A}$ if

- $X \cup Y = A$,
- $x \leq y$ for all $x \in X$ and all $y \in Y$,
- Y is closed under $\cdot$ and
- $x \cdot y = x$ for all $x \in X$ and all $y \in Y$.

$\mathbf{A}$ is called saturated if for every cut (X, Y) there exists $u \in \mathrm{Id}(\mathbf{A})$ such that $x \leq u \leq y$ for all $x \in X$ and all $y \in Y$.

Representation of saturated BL-chains

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- The Gödel chain $\bigoplus_{[0,1]} \text{Ł}_2$ is a saturated chain that is not representable.

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Lemma

Let $\mathbf{A} \cong \bigoplus_{i \in P} \mathbf{W}_i$ be a saturated BL-chain. If $\mathbf{W}_j$ is an unbounded hoop for some $j \in P$, then there exists $j_0 \in P$ preceding j such that $\mathbf{W}_{j_0} \cong \text{Ł}_2$.

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- The Gödel chain $\bigoplus_{[0,1]} \mathbf{Ł}_2$ is a saturated chain that is not representable.

Lemma

Let $\mathbf{A} \cong \bigoplus_{i \in P} \mathbf{W}_i$ be a saturated BL-chain. If $\mathbf{W}_j$ is an unbounded hoop for some $j \in P$, then there exists $j_0 \in P$ preceding j such that $\mathbf{W}_{j_0} \cong \mathbf{Ł}_2$.

Theorem

Let $\mathbf{A}$ be a saturated BL-chain and let $\bigoplus_{i \in P} \mathbf{W}_i$ be its unique decomposition into non-trivial Wajsberg hoops. If P is a well-ordered set, then there is a well-ordered set P' such that $\mathbf{A} \cong \bigoplus_{i \in P'} \mathbf{A}_i$, with $\mathbf{A}_i$ an MV-chain or a product chain. Consequently, $\mathbf{A}$ is ΠMV-representable.

Representation of saturated BL-chains

We know that $\mathbf{A} \cong \bigoplus_{i \in P} \mathbf{W}_i$ and P is a well-ordered set. As remarked, a hoop $\mathbf{W}_i$ in the decomposition of a BL-chain $\mathbf{A}$ can be unbounded. For instance, let us assume that $\mathbf{W}_j$ and $\mathbf{W}_k$ are unbounded hoops for some $j, k \in P$.

$$\mathbf{A} \cong \bigoplus_{i \in P} \mathbf{W}_i = \mathbf{W}_1 \oplus \ldots \oplus \mathbf{W}_j \oplus \ldots \oplus \mathbf{W}_k \oplus \ldots \oplus \mathbf{W}_l \oplus \ldots$$

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Let $j_0, k_0 \in P$ be the elements below j and k, respectively. Then

$$\mathbf{A} \cong \mathbf{W}_1 \oplus \ldots \oplus (\mathbf{W}_{j_0} \oplus \mathbf{W}_j) \oplus \ldots \oplus (\mathbf{W}_{k_0} \oplus \mathbf{W}_k) \oplus \ldots \oplus \mathbf{W}_l \oplus \ldots$$

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Moreover, since $\mathbf{W}_{j_0} \cong \mathbf{W}_{k_0} \cong \text{Ł}_2$,

$$\mathbf{A} \cong \mathbf{W}_1 \oplus \ldots \oplus (\text{Ł}_2 \oplus \mathbf{W}_j) \oplus \ldots \oplus (\text{Ł}_2 \oplus \mathbf{W}_k) \oplus \ldots \oplus \mathbf{W}_l \oplus \ldots$$

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Following the above suggested idea we define P' as a rearrangement of P. P' will index the summands

$$\mathbf{A}_i = \begin{cases} \text{Ł}_2 \oplus \mathbf{W}_i & \text{if } \mathbf{W}_i \text{ is unbounded;} \\ \mathbf{W}_i & \text{if } \mathbf{W}_i \text{ is bounded.} \end{cases}$$

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Then $\mathbf{A} \cong \bigoplus_{i \in P'} \mathbf{A}_i$ and each summand is an MV-chain or a product chain. Note that P' is a well-ordered set because P so is. Thus

$$\mathbf{A} \cong \bigotimes_{i \in P'} \mathbf{A}_i.$$

Representation of saturated BL-chains

The next result provides an alternative definition for ΠMV-representability. It also reveals the link between the notions of representability and ΠMV-representability.

Corollary

A BL-chain $\mathbf{A}$ is representable and saturated if and only if it is ΠMV-representable.

Further readings on the poset product construction

Busaniche, M., and F. Montagna, 'Hájek's logic BL and BL-algebras', in *Handbook of Mathematical Fuzzy Logic*, vol. 1 of *Studies in Logic, Mathematical Logic and Foundations*, chap. V, College Publications, London, 2011, pp. 355–447.

Jipsen, P., *'Generalizations of boolean products for lattice-ordered algebras'*, *Annals of Pure and Applied Logic*, 161 (2009), 228–234

Jipsen, P., and F. Montagna, 'On the structure of generalized BL-algebras', *Algebra Universalis*, 55 (2006), 227–238.

Jipsen, P., and F. Montagna, 'The Blok-Ferreirim theorem for normal GBL-algebras and its applications', *Algebra Universalis*, 60 (2009), 381–404.

Jipsen, P., and F. Montagna, 'Embedding theorems for classes of GBL-algebras', *Journal of Pure and Applied Algebra*, 214 (2010), 1559–1575.

Thank you