Hypersequents & Systems of Rules: An Embedding

Agata Ciabattoni and Francesco A. Genco[1]

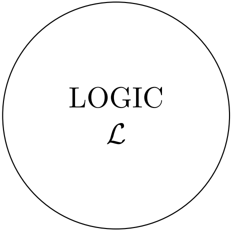
Syntax Meets Semantics 2016, Barcelona

FAKULTÄT FÜR !NFORMATIK
Faculty of Informatics

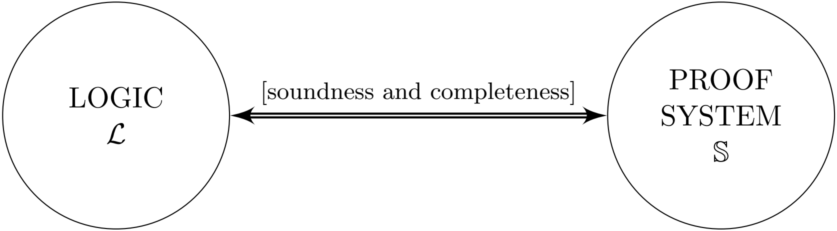
logics Logical Methods in Computer Science

[1]Funded by FWF project W1255-N23.

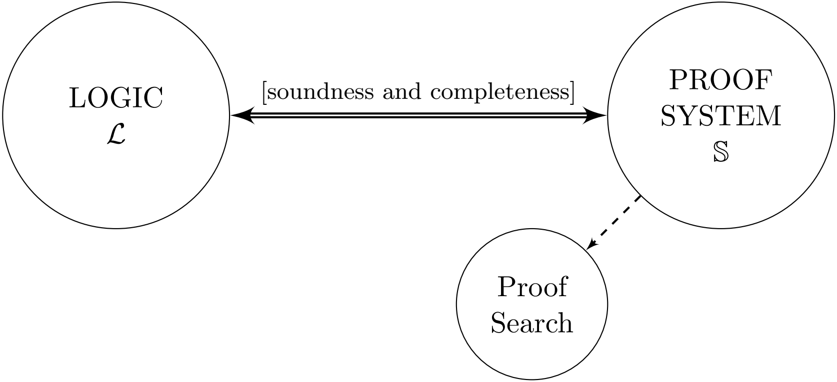
Logics, proof systems and formalisms



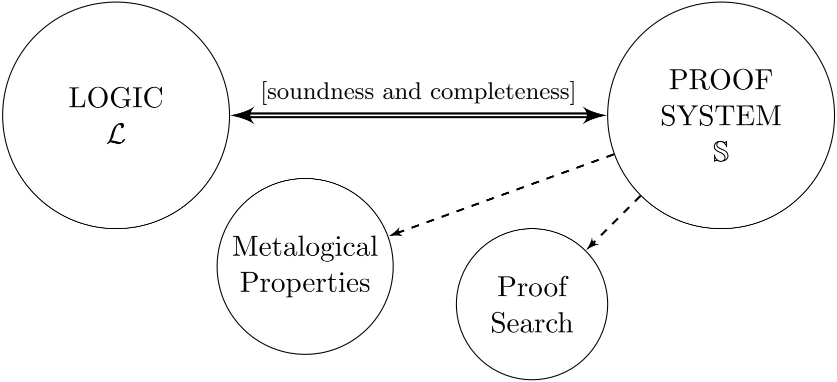
Logics, proof systems and formalisms



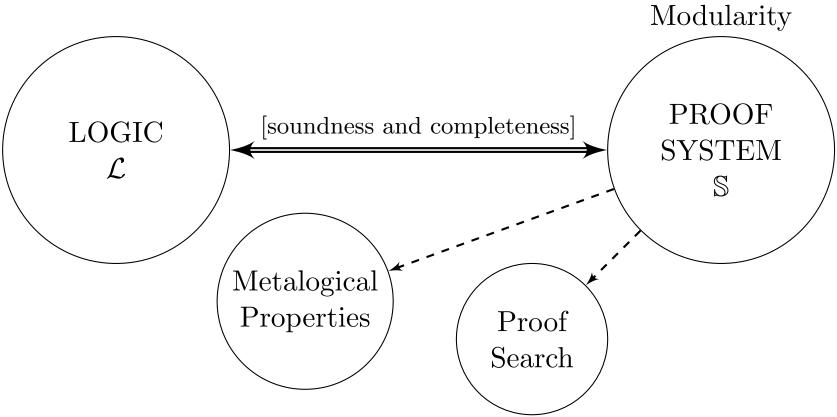
Logics, proof systems and formalisms



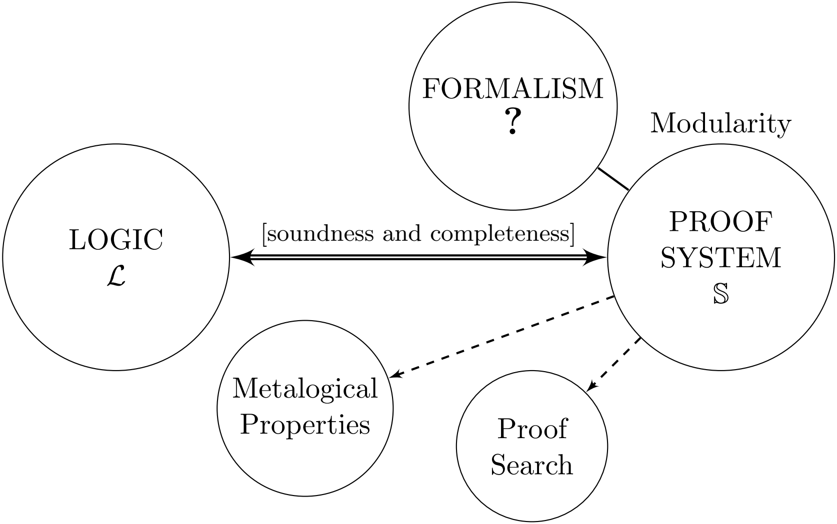
Logics, proof systems and formalisms



Logics, proof systems and formalisms



Logics, proof systems and formalisms



Good proof systems

ANALYTICITY

Derivations of a formula A only contain formulae which are subformulae of A

Good proof systems

ANALYTICITY

Derivations of a formula A only contain formulae which are subformulae of A

MODULARITY

For a logic $\mathcal{L} + axioms$ we need:

1. base system $\mathbb{S}$ for $\mathcal{L}$
2. some (hopefully) analytic rules for the *axioms*

Adding one axiom to $\mathcal{L}$ $\Rightarrow$ Adding some rules to $\mathbb{S}$

A jungle of formalisms

Many formalisms to capture logics

(sequents, hypersequents, labelled sequents,
nested sequents, display calculus, calculus of structures…)

A jungle of formalisms

Many formalisms to capture logics

(sequents, hypersequents, labelled sequents,
nested sequents, display calculus, calculus of structures…)

Embeddings between formalisms

$\Downarrow$

- Expressiveness relations
- Transfer of results
  (avoiding repetitions and possible mistakes)

[Wansing, 1998], [Fitting, 2012],
[Goré and Ramanayake, 2012], [Ramanayake 2015, 2016]…

Our context

To obtain **analytic** and **modular** (sequent-like) proof systems:

Extend the base proof system
by **analyticity preserving structural rules**
—those not referring to connectives—

$\Downarrow$

Modularity (fixed logical rules)

General translation methods from axioms to rules
[http://www.logic.at/people/lara/axiomcalc.html]

Sequents and beyond

Sequents are **simple** and **versatile**

$$\Gamma \Rightarrow \Delta$$

Sequents and beyond

Sequents are **simple** and **versatile**

$$\Gamma \Rightarrow \Delta$$

but **not enough** to define modular analytic proof systems **for many interesting logics**

Consider the axioms for **intermediate logics**:

$$\begin{cases} \neg\neg A \vee \neg A & \textit{Jankov} \\ (A \supset B) \vee (B \supset A) & \textit{Gödel} \\ A \vee (A \supset (B \vee (B \supset C))) & Bd_2 \\ \vdots & \end{cases}$$

No sequent structural rule can capture these axioms [Ciabattoni et al., 2012, Ann. Pure Appl. Logic]

So what? More structure!

The ***linearity axiom*** characterises *Gödel logic*

$$(A \supset B) \lor (B \supset A)$$

So what? More structure!

The ***linearity axiom*** characterises *Gödel logic*

$$(A \supset B) \vee (B \supset A)$$

We can define structural rules based on the syntax of this axiom using two (simple) generalisations of sequents:

HYPERSEQUENTS
[Mints, 1968]
[Pottinger, 1983]
[Avron, 1987]

and

SYSTEMS OF RULES
[Negri, 2014]

Hypersequents [Mints, 1968], [Pottinger, 1983], [Avron, 1987]

$$\frac{\text{Multisets of sequents (interpreted disjunctively)}}{\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n}$$

Hypersequents [Mints, 1968], [Pottinger, 1983], [Avron, 1987]

$$\frac{\text{Multisets of sequents (interpreted disjunctively)}}{\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n}$$

We can represent the *linearity axiom* as

$$(A \supset B) \vee (B \supset A)$$

Hypersequents [Mints, 1968], [Pottinger, 1983], [Avron, 1987]

Multisets of sequents (interpreted disjunctively)

$$\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$$

We can represent the *linearity axiom* as

$$\Rightarrow A \supset B \mid \Rightarrow B \supset A$$

Hypersequents [Mints, 1968], [Pottinger, 1983], [Avron, 1987]

$$\frac{\text{Multisets of sequents (interpreted disjunctively)}}{\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n}$$

We can represent the *linearity axiom* as

$$A \Rightarrow B \mid B \Rightarrow A$$

Hypersequents [Mints, 1968], [Pottinger, 1983], [Avron, 1987]

$$\frac{\text{Multisets of sequents (interpreted disjunctively)}}{\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n}$$

We can represent the *linearity axiom* as

$$A \Rightarrow B \mid B \Rightarrow A$$

and transform this into the rule

$$\frac{\mathcal{G} \mid B, \Gamma_1 \Rightarrow \Delta_1 \qquad \mathcal{G} \mid A, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid A, \Gamma_1 \Rightarrow \Delta_1 \mid B, \Gamma_2 \Rightarrow \Delta_2}$$

Example of hypersequent derivation

$$\frac{\mathcal{G} \mid B, \Gamma_1 \Rightarrow \Delta_1 \qquad \mathcal{G} \mid A, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid A, \Gamma_1 \Rightarrow \Delta_1 \mid B, \Gamma_2 \Rightarrow \Delta_2}\ (com) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ (EC)$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{B \Rightarrow B}\ init. \qquad \cfrac{}{A \Rightarrow A}\ init.}{A \Rightarrow B \mid B \Rightarrow A}\ (com)}{A \Rightarrow B \mid \Rightarrow B \supset A}\ (\supset r)}{\Rightarrow A \supset B \mid \Rightarrow B \supset A}\ (\supset r)}{\Rightarrow A \supset B \mid \Rightarrow (A \supset B) \vee (B \supset A)}\ (\vee r)}{\Rightarrow (A \supset B) \vee (B \supset A) \mid \Rightarrow (A \supset B) \vee (B \supset A)}\ (\vee r)}{\Rightarrow (A \supset B) \vee (B \supset A)}\ (EC)$$

Example of hypersequent derivation

$$\frac{\mathcal{G} \mid B, \Gamma_1 \Rightarrow \Delta_1 \qquad \mathcal{G} \mid A, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid A, \Gamma_1 \Rightarrow \Delta_1 \mid B, \Gamma_2 \Rightarrow \Delta_2}\ (com) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ (EC)$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{B \Rightarrow B}\ init. \qquad \cfrac{}{A \Rightarrow A}\ init.}{A \Rightarrow B \mid B \Rightarrow A}\ (com)}{A \Rightarrow B \mid \Rightarrow B \supset A}\ (\supset r)}{\Rightarrow A \supset B \mid \Rightarrow B \supset A}\ (\supset r)}{\Rightarrow A \supset B \mid \Rightarrow (A \supset B) \vee (B \supset A)}\ (\vee r)}{\Rightarrow (A \supset B) \vee (B \supset A) \mid \Rightarrow (A \supset B) \vee (B \supset A)}\ (\vee r)}{\Rightarrow (A \supset B) \vee (B \supset A)}\ (EC)$$

Example of hypersequent derivation

$$\frac{\mathcal{G} \mid B, \Gamma_1 \Rightarrow \Delta_1 \qquad \mathcal{G} \mid A, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid A, \Gamma_1 \Rightarrow \Delta_1 \mid B, \Gamma_2 \Rightarrow \Delta_2}\ (com) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ (EC)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow B}\ init. \qquad \dfrac{}{A \Rightarrow A}\ init.}{A \Rightarrow B \mid B \Rightarrow A}\ (com)}{A \Rightarrow B \mid \Rightarrow B \supset A}\ (\supset r)}{\Rightarrow A \supset B \mid \Rightarrow B \supset A}\ (\supset r)}{\Rightarrow A \supset B \mid \Rightarrow (A \supset B) \vee (B \supset A)}\ (\vee r)}{\Rightarrow (A \supset B) \vee (B \supset A) \mid \Rightarrow (A \supset B) \vee (B \supset A)}\ (\vee r)}{\Rightarrow (A \supset B) \vee (B \supset A)}\ (EC)$$

Example of hypersequent derivation

$$\frac{\mathcal{G} \mid B, \Gamma_1 \Rightarrow \Delta_1 \qquad \mathcal{G} \mid A, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid A, \Gamma_1 \Rightarrow \Delta_1 \mid B, \Gamma_2 \Rightarrow \Delta_2} \; (com) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta} \; (EC)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow B}\,init. \qquad \dfrac{}{A \Rightarrow A}\,init.}{A \Rightarrow B \mid {\color{red} B \Rightarrow A}}\,(com)}{{\color{red} A \Rightarrow B} \mid \Rightarrow B \supset A}\,(\supset r)}{\Rightarrow A \supset B \mid \Rightarrow B \supset A}\,(\supset r)}{\Rightarrow A \supset B \mid \Rightarrow (A \supset B) \vee (B \supset A)}\,(\vee r)}{\Rightarrow (A \supset B) \vee (B \supset A) \mid \Rightarrow (A \supset B) \vee (B \supset A)}\,(\vee r)}{\Rightarrow (A \supset B) \vee (B \supset A)}\,(EC)$$

Example of hypersequent derivation

$$\frac{\mathcal{G} \mid B, \Gamma_1 \Rightarrow \Delta_1 \qquad \mathcal{G} \mid A, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid A, \Gamma_1 \Rightarrow \Delta_1 \mid B, \Gamma_2 \Rightarrow \Delta_2}\ (com) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ (EC)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow B}\ init. \qquad \dfrac{}{A \Rightarrow A}\ init.}{A \Rightarrow B \mid B \Rightarrow A}\ (com)}{A \Rightarrow B \mid \Rightarrow B \supset A}\ (\supset r)}{\Rightarrow A \supset B \mid \Rightarrow B \supset A}\ (\supset r)}{\Rightarrow A \supset B \mid \Rightarrow (A \supset B) \vee (B \supset A)}\ (\vee r)}{\Rightarrow (A \supset B) \vee (B \supset A) \mid \Rightarrow (A \supset B) \vee (B \supset A)}\ (\vee r)}{\Rightarrow (A \supset B) \vee (B \supset A)}\ (EC)$$

Example of hypersequent derivation

$$\frac{\mathcal{G} \mid B, \Gamma_1 \Rightarrow \Delta_1 \qquad \mathcal{G} \mid A, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid A, \Gamma_1 \Rightarrow \Delta_1 \mid B, \Gamma_2 \Rightarrow \Delta_2}\ (com) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ (EC)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow B}\ init. \qquad \dfrac{}{A \Rightarrow A}\ init.}{A \Rightarrow B \mid B \Rightarrow A}\ (com)}{A \Rightarrow B \mid \Rightarrow B \supset A}\ (\supset r)}{\Rightarrow A \supset B \mid \Rightarrow B \supset A}\ (\supset r)}{\Rightarrow A \supset B \mid \Rightarrow (A \supset B) \vee (B \supset A)}\ (\vee r)}{\Rightarrow (A \supset B) \vee (B \supset A) \mid \Rightarrow (A \supset B) \vee (B \supset A)}\ (\vee r)}{\Rightarrow (A \supset B) \vee (B \supset A)}\ (EC)$$

Example of hypersequent derivation

$$\frac{\mathcal{G} \mid B, \Gamma_1 \Rightarrow \Delta_1 \qquad \mathcal{G} \mid A, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid A, \Gamma_1 \Rightarrow \Delta_1 \mid B, \Gamma_2 \Rightarrow \Delta_2}\ (com) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \Delta}\ (EC)$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{B \Rightarrow B}\ init. \qquad \cfrac{}{A \Rightarrow A}\ init.}{A \Rightarrow B \mid B \Rightarrow A}\ (com)}{A \Rightarrow B \mid \Rightarrow B \supset A}\ (\supset r)}{\Rightarrow A \supset B \mid \Rightarrow B \supset A}\ (\supset r)}{\Rightarrow A \supset B \mid \Rightarrow (A \supset B) \vee (B \supset A)}\ (\vee r)}{\Rightarrow (A \supset B) \vee (B \supset A) \mid \Rightarrow (A \supset B) \vee (B \supset A)}\ (\vee r)}{\Rightarrow (A \supset B) \vee (B \supset A)}\ (EC)$$

Systems of rules [Negri, 2014, J. Logic Comput]

Sets of rules related by applicability constraints

Systems of rules

Sets of rules related by applicability constraints

We will only consider
purely syntactical and **two-level systems**:

$$\dfrac{\Gamma_1^1 \Rightarrow \Delta_1^1 \;\ldots\; \Gamma_1^{n_1} \Rightarrow \Delta_1^{n_1}}{\Gamma_1 \Rightarrow \Delta_1}\,(top_1) \qquad\qquad \dfrac{\Gamma_k^1 \Rightarrow \Delta_k^1 \;\ldots\; \Gamma_k^{n_k} \Rightarrow \Delta_k^{n_k}}{\Gamma_k \Rightarrow \Delta_k}\,(top_k)$$

$$\vdots \qquad\qquad\qquad\qquad\qquad \vdots$$

$$\dfrac{\Gamma \Rightarrow \Delta \qquad\qquad \ldots \qquad\qquad \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,(bottom)$$

Example of system

We represent the axiom $(A \supset B) \vee (B \supset A)$ as the system

$$\dfrac{\dfrac{\dfrac{B, \Gamma_1 \Rightarrow \Delta_1}{A, \Gamma_1 \Rightarrow \Delta_1}\ (com_1)}{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta\end{array}} \qquad \dfrac{\dfrac{A, \Gamma_2 \Rightarrow \Delta_2}{B, \Gamma_2 \Rightarrow \Delta_2}\ (com_2)}{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta\end{array}}}{\Gamma \Rightarrow \Delta}\ (end)$$

Example of systems of rules derivation

$$\dfrac{\dfrac{\dfrac{B, \Gamma_1 \Rightarrow \Delta_1}{A, \Gamma_1 \Rightarrow \Delta_1}\ (com_1)}{\vdots}\qquad \dfrac{\dfrac{A, \Gamma_2 \Rightarrow \Delta_2}{B, \Gamma_2 \Rightarrow \Delta_2}\ (com_1)}{\vdots}}{}$$

$$\dfrac{\Gamma \Rightarrow \Delta \qquad\qquad \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (end)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow B}\ init.}{A \Rightarrow B}\ (com_1)}{\Rightarrow A \supset B}\ (\supset r)}{\Rightarrow (A \supset B) \vee (B \supset A)}\ (\vee r) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\ init.}{B \Rightarrow A}\ (com_2)}{\Rightarrow B \supset A}\ (\supset r)}{\Rightarrow (A \supset B) \vee (B \supset A)}\ (\vee r)}{\Rightarrow (A \supset B) \vee (B \supset A)}\ (end)$$

Example of systems of rules derivation

$$
\dfrac{
\begin{array}{c}
\dfrac{B, \Gamma_1 \Rightarrow \Delta_1}{A, \Gamma_1 \Rightarrow \Delta_1}\ (com_1) \\
\vdots \\
\color{red}{\Gamma \Rightarrow \Delta}
\end{array}
\qquad\qquad
\begin{array}{c}
\dfrac{A, \Gamma_2 \Rightarrow \Delta_2}{B, \Gamma_2 \Rightarrow \Delta_2}\ (com_1) \\
\vdots \\
\color{red}{\Gamma \Rightarrow \Delta}
\end{array}
}{\Gamma \Rightarrow \Delta}\ (end)
$$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{B \Rightarrow B}\ init.
}{A \Rightarrow B}\ (com_1)
}{\Rightarrow A \supset B}\ (\supset r)
}{\color{red}{\Rightarrow (A \supset B) \vee (B \supset A)}}\ (\vee r)
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{A \Rightarrow A}\ init.
}{B \Rightarrow A}\ (com_2)
}{\Rightarrow B \supset A}\ (\supset r)
}{\color{red}{\Rightarrow (A \supset B) \vee (B \supset A)}}\ (\vee r)
}{\Rightarrow (A \supset B) \vee (B \supset A)}\ (end)
$$

Example of systems of rules derivation

$$\dfrac{\dfrac{\dfrac{B, \Gamma_1 \Rightarrow \Delta_1}{A, \Gamma_1 \Rightarrow \Delta_1}\,(com_1)}{\vdots \atop \Gamma \Rightarrow \Delta} \qquad \dfrac{\dfrac{A, \Gamma_2 \Rightarrow \Delta_2}{B, \Gamma_2 \Rightarrow \Delta_2}\,(com_1)}{\vdots \atop \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}\,(end)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow B}\,init.}{A \Rightarrow B}\,(com_1)}{\Rightarrow A \supset B}\,(\supset r)}{\Rightarrow (A \supset B) \vee (B \supset A)}\,(\vee r) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,init.}{B \Rightarrow A}\,(com_2)}{\Rightarrow B \supset A}\,(\supset r)}{\Rightarrow (A \supset B) \vee (B \supset A)}\,(\vee r)}{\Rightarrow (A \supset B) \vee (B \supset A)}\,(end)$$

Example of systems of rules derivation

$$
\dfrac{\dfrac{\dfrac{B, \Gamma_1 \Rightarrow \Delta_1}{A, \Gamma_1 \Rightarrow \Delta_1}\,(com_1)}{\vdots \atop \Gamma \Rightarrow \Delta} \qquad \dfrac{\dfrac{A, \Gamma_2 \Rightarrow \Delta_2}{B, \Gamma_2 \Rightarrow \Delta_2}\,(com_1)}{\vdots \atop \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}\,(end)
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow B}\,init.}{A \Rightarrow B}\,(com_1)}{\Rightarrow A \supset B}\,(\supset r)}{\Rightarrow (A \supset B) \vee (B \supset A)}\,(\vee r) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,init.}{B \Rightarrow A}\,(com_2)}{\Rightarrow B \supset A}\,(\supset r)}{\Rightarrow (A \supset B) \vee (B \supset A)}\,(\vee r)}{\Rightarrow (A \supset B) \vee (B \supset A)}\,(end)
$$

Example of systems of rules derivation

$$
\dfrac{\dfrac{\color{red}{B}, \Gamma_1 \Rightarrow \Delta_1}{A, \Gamma_1 \Rightarrow \Delta_1}\,(com_1) \quad \vdots \quad \Gamma \Rightarrow \Delta \qquad\qquad \dfrac{A, \Gamma_2 \Rightarrow \Delta_2}{\color{red}{B}, \Gamma_2 \Rightarrow \Delta_2}\,(com_1) \quad \vdots \quad \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,(end)
$$

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{}{\color{red}{B} \Rightarrow B}\,init.}{A \Rightarrow B}\,(com_1)}{\Rightarrow A \supset B}\,(\supset r)}{\Rightarrow (A \supset B) \vee (B \supset A)}\,(\vee r)
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,init.}{\color{red}{B} \Rightarrow A}\,(com_2)}{\Rightarrow B \supset A}\,(\supset r)}{\Rightarrow (A \supset B) \vee (B \supset A)}\,(\vee r)
}{\Rightarrow (A \supset B) \vee (B \supset A)}\,(end)
$$

Example of systems of rules derivation

$$
\dfrac{\dfrac{\dfrac{B, \Gamma_1 \Rightarrow \Delta_1}{A, \Gamma_1 \Rightarrow \Delta_1}\ (com_1)}{\vdots \atop \Gamma \Rightarrow \Delta} \qquad \dfrac{\dfrac{A, \Gamma_2 \Rightarrow \Delta_2}{B, \Gamma_2 \Rightarrow \Delta_2}\ (com_1)}{\vdots \atop \Gamma \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}\ (end)
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow B}\ init.}{A \Rightarrow B}\ (com_1)}{\Rightarrow A \supset B}\ (\supset r)}{\Rightarrow (A \supset B) \vee (B \supset A)}\ (\vee r) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\ init.}{B \Rightarrow A}\ (com_2)}{\Rightarrow B \supset A}\ (\supset r)}{\Rightarrow (A \supset B) \vee (B \supset A)}\ (\vee r)}{\Rightarrow (A \supset B) \vee (B \supset A)}\ (end)
$$

A casual resemblance?

$$\frac{\mathcal{G} \mid B, \Gamma_1 \Rightarrow \Delta_1 \qquad \mathcal{G} \mid A, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid A, \Gamma_1 \Rightarrow \Delta_1 \mid B, \Gamma_2 \Rightarrow \Delta_2}$$

$$\dfrac{\begin{array}{c}\dfrac{B, \Gamma_1 \Rightarrow \Delta_1}{A, \Gamma_1 \Rightarrow \Delta_1} \\ \vdots \\ \Gamma \Rightarrow \Delta\end{array} \qquad \begin{array}{c}\dfrac{A, \Gamma_2 \Rightarrow \Delta_2}{B, \Gamma_2 \Rightarrow \Delta_2} \\ \vdots \\ \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}$$

The two formalisms seem to be related

A possible connection is suggested in
[Negri, 2014, J. Logic Comput.]

A casual resemblance?

$$\frac{\mathcal{G} \mid B, \Gamma_1 \Rightarrow \Delta_1 \qquad \mathcal{G} \mid A, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid A, \Gamma_1 \Rightarrow \Delta_1 \mid B, \Gamma_2 \Rightarrow \Delta_2}$$

$$\dfrac{\begin{array}{c}\dfrac{B, \Gamma_1 \Rightarrow \Delta_1}{A, \Gamma_1 \Rightarrow \Delta_1} \\ \vdots \\ \Gamma \Rightarrow \Delta\end{array} \qquad \begin{array}{c}\dfrac{A, \Gamma_2 \Rightarrow \Delta_2}{B, \Gamma_2 \Rightarrow \Delta_2} \\ \vdots \\ \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}$$

The two formalisms seem to be related

A possible connection is suggested in [Negri, 2014, J. Logic Comput.]

Can we fully formalise this intuition?

Where does this lead to?

Our Embedding

A. Ciabattoni and F. A. Genco. *Embedding formalisms: hypersequents and two-level systems of rules. Advances in Modal Logic 2016.*

Rules translation

- Any hypersequent rule can be rewritten as a two-level system of rules
- Any two-level system of rules can be rewritten as a hypersequent rule

$$\frac{\mathcal{G} \mid \Gamma_1' \Rightarrow \Delta_1' \qquad \dots \qquad \mathcal{G} \mid \Gamma_k' \Rightarrow \Delta_k'}{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n}$$

$\mathcal{S}_1, \dots, \mathcal{S}_n$ sets of sequents $\Downarrow$ $\mathcal{S}_1 \cup \dots \cup \mathcal{S}_n = \{\Gamma_i' \Rightarrow \Delta_i'\}_{1 \leq i \leq k}$

$$\frac{\begin{array}{c}\dfrac{\mathcal{S}_1}{\Gamma_1 \Rightarrow \Delta_1} \\ \vdots \\ \Gamma \Rightarrow \Delta\end{array} \qquad \cdots \qquad \begin{array}{c}\dfrac{\mathcal{S}_n}{\Gamma_n \Rightarrow \Delta_n} \\ \vdots \\ \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}$$

Rules translation

- Any hypersequent rule can be rewritten as a two-level system of rules
- Any two-level system of rules can be rewritten as a hypersequent rule

$$\frac{\mathcal{G} \mid \Gamma'_1 \Rightarrow \Delta'_1 \qquad \dots \qquad \mathcal{G} \mid \Gamma'_k \Rightarrow \Delta'_k}{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n}$$

$\mathcal{S}_1, \dots, \mathcal{S}_n$ sets of sequents $\quad \Uparrow \quad$ $\mathcal{S}_1 \cup \dots \cup \mathcal{S}_n = \{\Gamma'_i \Rightarrow \Delta'_i\}_{1 \leq i \leq k}$

$$\frac{\begin{array}{c}\dfrac{\mathcal{S}_1}{\Gamma_1 \Rightarrow \Delta_1} \\ \vdots \\ \Gamma \Rightarrow \Delta\end{array} \qquad \cdots \qquad \begin{array}{c}\dfrac{\mathcal{S}_n}{\Gamma_n \Rightarrow \Delta_n} \\ \vdots \\ \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}$$

Derivations translation

- Any hypersequent derivation can be translated into a two-level systems of rules derivation
- Any two-level systems of rules derivation can be translated into a hypersequent derivation

Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A \Rightarrow A \mid B \Rightarrow A \wedge B}\,(EW) \qquad \dfrac{\dfrac{B \Rightarrow B \quad A \Rightarrow A}{A \Rightarrow B \mid B \Rightarrow A}\,(com) \qquad \dfrac{B \Rightarrow B}{A \Rightarrow B \mid B \Rightarrow B}\,(EW)}{A \Rightarrow B \mid B \Rightarrow A \wedge B}\,(\wedge r)}{A \Rightarrow A \wedge B \mid B \Rightarrow A \wedge B}\,(\wedge r)}{A \Rightarrow A \wedge B \mid \Rightarrow B \supset (A \wedge B)}\,(\supset r)}{\Rightarrow A \supset (A \wedge B) \mid \Rightarrow B \supset (A \wedge B)}\,(\supset r)}{\Rightarrow A \supset (A \wedge B) \mid \Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r)}{\dfrac{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B)) \mid \Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(EC)}\,(\vee r)
$$

$$\Updownarrow$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \qquad \dfrac{B \Rightarrow B}{A \Rightarrow B}\,(com_1)}{A \Rightarrow A \wedge B}\,(\wedge r)}{\Rightarrow A \supset (A \wedge B)}\,(\supset r)}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r) \qquad \dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{B \Rightarrow A}\,(com_2) \qquad B \Rightarrow B}{B \Rightarrow A \wedge B}\,(\wedge r)}{\Rightarrow B \supset (A \wedge B)}\,(\supset r)}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r)}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(end)
$$

Example

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{A \Rightarrow A}{A \Rightarrow A \mid B \Rightarrow A \wedge B}\,(EW)
          \qquad
          \dfrac{
            \dfrac{B \Rightarrow B \quad A \Rightarrow A}{A \Rightarrow B \mid B \Rightarrow A}\,(com)
            \quad
            \dfrac{B \Rightarrow B}{A \Rightarrow B \mid B \Rightarrow B}\,(EW)
          }{A \Rightarrow B \mid B \Rightarrow A \wedge B}\,(\wedge r)
        }{A \Rightarrow A \wedge B \mid B \Rightarrow A \wedge B}\,(\wedge r)
      }{A \Rightarrow A \wedge B \mid \Rightarrow B \supset (A \wedge B)}\,(\supset r)
    }{\Rightarrow A \supset (A \wedge B) \mid \Rightarrow B \supset (A \wedge B)}\,(\supset r)
  }{\Rightarrow A \supset (A \wedge B) \mid \Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r)
}{
  \dfrac{{\color{red}\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))} \mid {\color{blue}\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(EC)
}\,(\vee r)
$$

$${\color{red}\Updownarrow}$$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{A \Rightarrow A \quad \dfrac{B \Rightarrow B}{A \Rightarrow B}\,(com_1)}{A \Rightarrow A \wedge B}\,(\wedge r)
    }{\Rightarrow A \supset (A \wedge B)}\,(\supset r)
  }{{\color{red}\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}}\,(\vee r)
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{\dfrac{A \Rightarrow A}{B \Rightarrow A}\,(com_2) \quad B \Rightarrow B}{B \Rightarrow A \wedge B}\,(\wedge r)
    }{\Rightarrow B \supset (A \wedge B)}\,(\supset r)
  }{{\color{blue}\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}}\,(\vee r)
}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(end)
$$

Example

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{A \Rightarrow A}{A \Rightarrow A \mid B \Rightarrow A \wedge B}\,(EW)
\qquad
\dfrac{
\dfrac{B \Rightarrow B \quad A \Rightarrow A}{A \Rightarrow B \mid B \Rightarrow A}\,(com)
\qquad
\dfrac{B \Rightarrow B}{A \Rightarrow B \mid B \Rightarrow B}\,(EW)
}{A \Rightarrow B \mid B \Rightarrow A \wedge B}\,(\wedge r)
}{A \Rightarrow A \wedge B \mid B \Rightarrow A \wedge B}\,(\wedge r)
}{A \Rightarrow A \wedge B \mid \Rightarrow B \supset (A \wedge B)}\,(\supset r)
}{\Rightarrow A \supset (A \wedge B) \mid \Rightarrow B \supset (A \wedge B)}\,(\supset r)
}{\Rightarrow A \supset (A \wedge B) \mid \Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r)
}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B)) \mid \Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r)
}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(EC)
$$

$$\Updownarrow$$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{A \Rightarrow A \quad \dfrac{B \Rightarrow B}{A \Rightarrow B}\,(com_1)}{A \Rightarrow A \wedge B}\,(\wedge r)
}{\Rightarrow A \supset (A \wedge B)}\,(\supset r)
}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r)
\qquad
\dfrac{
\dfrac{
\dfrac{\dfrac{A \Rightarrow A}{B \Rightarrow A}\,(com_2) \quad B \Rightarrow B}{B \Rightarrow A \wedge B}\,(\wedge r)
}{\Rightarrow B \supset (A \wedge B)}\,(\supset r)
}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r)
}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(end)
$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A \Rightarrow A \mid B \Rightarrow A \wedge B}\,(EW) \quad \dfrac{\dfrac{B \Rightarrow B \quad A \Rightarrow A}{A \Rightarrow B \mid B \Rightarrow A}\,(com) \quad \dfrac{B \Rightarrow B}{A \Rightarrow B \mid B \Rightarrow B}\,(EW)}{A \Rightarrow B \mid B \Rightarrow A \wedge B}\,(\wedge r)}{A \Rightarrow A \wedge B \mid {\color{blue} B \Rightarrow A \wedge B}}\,(\wedge r)}{{\color{red} A \Rightarrow A \wedge B} \mid\, \Rightarrow B \supset (A \wedge B)}\,(\supset r)}{\Rightarrow A \supset (A \wedge B) \mid\, \Rightarrow B \supset (A \wedge B)}\,(\supset r)}{\Rightarrow A \supset (A \wedge B) \mid\, \Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r)}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B)) \mid\, \Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r)}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(EC)$$

$$\color{red}\Updownarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \quad \dfrac{B \Rightarrow B}{A \Rightarrow B}\,(com_1)}{{\color{red} A \Rightarrow A \wedge B}}\,(\wedge r)}{\Rightarrow A \supset (A \wedge B)}\,(\supset r)}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r) \quad \dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{B \Rightarrow A}\,(com_2) \quad B \Rightarrow B}{{\color{blue} B \Rightarrow A \wedge B}}\,(\wedge r)}{\Rightarrow B \supset (A \wedge B)}\,(\supset r)}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r)}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(end)$$

Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A \Rightarrow A \mid B \Rightarrow A \wedge B}\,(EW) \qquad \dfrac{\dfrac{B \Rightarrow B \quad A \Rightarrow A}{A \Rightarrow B \mid B \Rightarrow A}\,(com) \qquad \dfrac{B \Rightarrow B}{A \Rightarrow B \mid B \Rightarrow B}\,(EW)}{A \Rightarrow B \mid B \Rightarrow A \wedge B}\,(\wedge r)}{A \Rightarrow A \wedge B \mid B \Rightarrow A \wedge B}\,(\wedge r)}{A \Rightarrow A \wedge B \mid\Rightarrow B \supset (A \wedge B)}\,(\supset r)}{\Rightarrow A \supset (A \wedge B) \mid \Rightarrow B \supset (A \wedge B)}\,(\supset r)}{\Rightarrow A \supset (A \wedge B) \mid\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r)}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B)) \mid \Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r)}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(EC)
$$

$$\Updownarrow$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \qquad \dfrac{B \Rightarrow B}{A \Rightarrow B}\,(com_1)}{A \Rightarrow A \wedge B}\,(\wedge r)}{\Rightarrow A \supset (A \wedge B)}\,(\supset r)}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r) \qquad \dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{B \Rightarrow A}\,(com_2) \qquad B \Rightarrow B}{B \Rightarrow A \wedge B}\,(\wedge r)}{\Rightarrow B \supset (A \wedge B)}\,(\supset r)}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r)}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(end)
$$

Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A \Rightarrow A \mid B \Rightarrow A \wedge B}\,(EW) \qquad \dfrac{\dfrac{B \Rightarrow B \quad A \Rightarrow A}{A \Rightarrow B \mid B \Rightarrow A}\,(com) \qquad \dfrac{B \Rightarrow B}{A \Rightarrow B \mid B \Rightarrow B}\,(EW)}{A \Rightarrow B \mid B \Rightarrow A \wedge B}\,(\wedge r)}{A \Rightarrow A \wedge B \mid B \Rightarrow A \wedge B}\,(\wedge r)}{A \Rightarrow A \wedge B \mid \Rightarrow B \supset (A \wedge B)}\,(\supset r)}{\Rightarrow A \supset (A \wedge B) \mid \Rightarrow B \supset (A \wedge B)}\,(\supset r)}{\Rightarrow A \supset (A \wedge B) \mid \Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r)}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B)) \mid \Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r)}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(EC)
$$

$$\Updownarrow$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \qquad \dfrac{B \Rightarrow B}{A \Rightarrow B}\,(com_1)}{A \Rightarrow A \wedge B}\,(\wedge r)}{\Rightarrow A \supset (A \wedge B)}\,(\supset r)}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r) \qquad \dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{B \Rightarrow A}\,(com_2) \qquad B \Rightarrow B}{B \Rightarrow A \wedge B}\,(\wedge r)}{\Rightarrow B \supset (A \wedge B)}\,(\supset r)}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r)}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(end)
$$

Example

$$
\dfrac{
\dfrac{A \Rightarrow A}{A \Rightarrow A \mid B \Rightarrow A \wedge B}\,(EW)
\qquad
\dfrac{
\dfrac{B \Rightarrow B \quad A \Rightarrow A}{A \Rightarrow B \mid B \Rightarrow A}\,(com)
\quad
\dfrac{B \Rightarrow B}{A \Rightarrow B \mid B \Rightarrow B}\,(EW)
}{A \Rightarrow B \mid B \Rightarrow A \wedge B}\,(\wedge r)
}{
\dfrac{
\dfrac{
\dfrac{
\dfrac{A \Rightarrow A \wedge B \mid B \Rightarrow A \wedge B}{A \Rightarrow A \wedge B \mid \Rightarrow B \supset (A \wedge B)}\,(\supset r)
}{\Rightarrow A \supset (A \wedge B) \mid \; \Rightarrow B \supset (A \wedge B)}\,(\supset r)
}{\Rightarrow A \supset (A \wedge B) \mid \Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r)
}{
\dfrac{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B)) \mid \; \Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(EC)
}\,(\vee r)
}\,(\wedge r)
$$

$$\Updownarrow$$

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{A \Rightarrow A \qquad \dfrac{B \Rightarrow B}{A \Rightarrow B}\,(com_1)}{A \Rightarrow A \wedge B}\,(\wedge r)
}{\Rightarrow A \supset (A \wedge B)}\,(\supset r)
}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r)
\qquad
\dfrac{
\dfrac{
\dfrac{\dfrac{A \Rightarrow A}{B \Rightarrow A}\,(com_2) \qquad B \Rightarrow B}{B \Rightarrow A \wedge B}\,(\wedge r)
}{\Rightarrow B \supset (A \wedge B)}\,(\supset r)
}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(\vee r)
}{\Rightarrow (A \supset (A \wedge B)) \vee (B \supset (A \wedge B))}\,(end)
$$

How far does this go?

The two formalisms are equivalent
w.r.t. intermediate logics, but…

The embedding only requires (EW), (EC), (EE)
and premisses with at most one active component

How far does this go?

The two formalisms are equivalent
w.r.t. intermediate logics, but…

The embedding only requires (EW), (EC), (EE)
and premisses with at most one active component

$\Downarrow$

can be naturally extended to other classes of propositional logics

Applications of the Embedding

Systems of rules made local

By the embedding we can
represent two-level systems of rules locally:

$$\dfrac{\dfrac{\dfrac{B, \Gamma_1 \Rightarrow \Delta_1}{A, \Gamma_1 \Rightarrow \Delta_1}}{\vdots}\quad \Gamma \Rightarrow \Delta \qquad \dfrac{\dfrac{A, \Gamma_2 \Rightarrow \Delta_2}{B, \Gamma_2 \Rightarrow \Delta_2}}{\vdots}\quad \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \quad \rightsquigarrow \quad \dfrac{\mathcal{G} \mid B, \Gamma_1 \Rightarrow \Delta_1 \qquad \mathcal{G} \mid A, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{G} \mid A, \Gamma_1 \Rightarrow \Delta_1 \mid B, \Gamma_2 \Rightarrow \Delta_2}$$

Systems of rules made analytic

In [Negri, 2014, J. Logic Comput.] it is proved that systems of rules acting on atoms preserve cut-elimination

Given **any** two-level system of rules we can:

1. Translate the system of rules into a hypersequent rule [embedding]

Systems of rules made analytic

In [Negri, 2014, J. Logic Comput.] it is proved that systems of rules acting on atoms preserve cut-elimination

Given **any** two-level system of rules we can:

1. Translate the system of rules into a hypersequent rule [embedding]
2. Apply the "completion" procedure [Ciabattoni et al., 2008, LICS]

Systems of rules made analytic

In [Negri, 2014, J. Logic Comput.] it is proved that
systems of rules acting on atoms preserve cut-elimination

Given **any** two-level system of rules we can:

1. Translate the system of rules into a hypersequent rule [embedding]
2. Apply the “completion” procedure [Ciabattoni et al., 2008, LICS]
3. Translate the result back [embedding]

Systems of rules made analytic

In [Negri, 2014, J. Logic Comput.] it is proved that systems of rules acting on atoms preserve cut-elimination

Given **any** two-level system of rules we can:

1. Translate the system of rules into a hypersequent rule [embedding]
2. Apply the “completion” procedure [Ciabattoni et al., 2008, LICS]
3. Translate the result back [embedding]

We thus obtain an **analytic 2-level system of rules**

(The embedding does not introduce new applications of the cut rule)

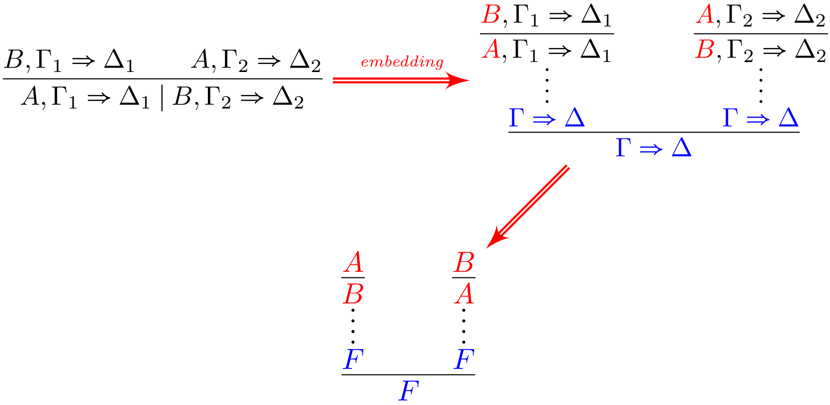
Hypersequents made natural

The embedding provides a reformulation
of hypersequent calculi into N.D.

$$\frac{B,\Gamma_1 \Rightarrow \Delta_1 \qquad A,\Gamma_2 \Rightarrow \Delta_2}{A,\Gamma_1 \Rightarrow \Delta_1 \mid B,\Gamma_2 \Rightarrow \Delta_2} \quad \xRightarrow{\textit{embedding}} \quad \dfrac{\begin{array}{c}\dfrac{B,\Gamma_1 \Rightarrow \Delta_1}{A,\Gamma_1 \Rightarrow \Delta_1}\\ \vdots \\ \Gamma \Rightarrow \Delta\end{array} \qquad \begin{array}{c}\dfrac{A,\Gamma_2 \Rightarrow \Delta_2}{B,\Gamma_2 \Rightarrow \Delta_2}\\ \vdots \\ \Gamma \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}$$

Hypersequents made natural

The embedding provides a reformulation
of hypersequent calculi into N.D.



Avron's conjecture

[Avron, 1991]

Intermediate logics formalised by **hypersequent** calculi could serve as base for **parallel λ-calculi**

Avron's conjecture

[Avron, 1991]

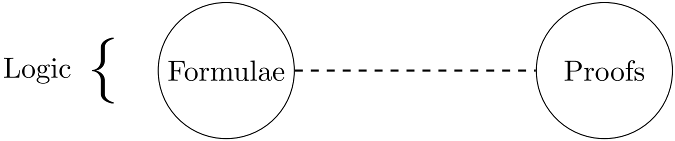
Intermediate logics formalised by **hypersequent** calculi could serve as base for **parallel λ-calculi**

<u>The Problem</u>

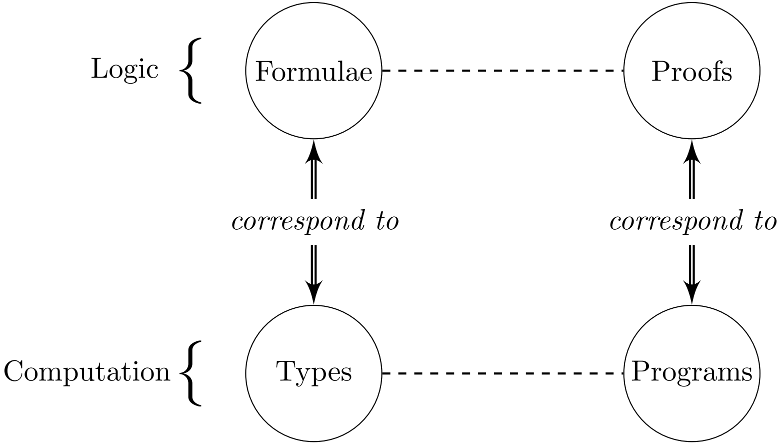
Find **computational interpretations** for these logics

$$\begin{array}{ccccc}
\text{Logic} & \Longrightarrow & \text{Hypersequents} & \xrightarrow{\textit{Embedding}} & \text{Systems of Rules} \\
\Uparrow & & & & \Downarrow \\
\lambda\text{-Calculus} & & \xleftarrow{\textit{Curry–Howard Correspondence}} & & \text{N.D.}
\end{array}$$

The Curry–Howard correspondence



The Curry–Howard correspondence



The Curry–Howard correspondence II

$$\dfrac{\begin{array}{c}\vdots \\ t : A \supset B\end{array} \qquad \begin{array}{c}\vdots \\ u : A\end{array}}{tu : B} \qquad\qquad \dfrac{\begin{array}{c}[x : A] \\ \vdots \\ t : B\end{array}}{\lambda x\, t : A \supset B}$$

The Curry–Howard correspondence II

$$\dfrac{\begin{matrix}\vdots \\ t : A \supset B\end{matrix} \qquad \begin{matrix}\vdots \\ u : A\end{matrix}}{tu : B} \qquad\qquad \dfrac{\begin{matrix}[x : A] \\ \vdots \\ t : B\end{matrix}}{\lambda x\, t : A \supset B}$$

$$\dfrac{\begin{matrix}\vdots \\ \dfrac{A}{B} \\ \vdots \\ F\end{matrix} \qquad\qquad \begin{matrix}\vdots \\ \dfrac{B}{A} \\ \vdots \\ F\end{matrix}}{F}$$

The Curry–Howard correspondence II

$$\dfrac{\begin{array}{c}\vdots\\ t : A \supset B\end{array} \qquad \begin{array}{c}\vdots\\ u : A\end{array}}{tu : B} \qquad\qquad \dfrac{\begin{array}{c}[x : A]\\ \vdots\\ t : B\end{array}}{\lambda x\, t : A \supset B}$$

$$\dfrac{\begin{array}{c}\vdots\\ \dfrac{s : A}{cs : B}\\ \vdots\\ u : F\end{array} \qquad \begin{array}{c}\vdots\\ \dfrac{t : B}{ct : A}\\ \vdots\\ v : F\end{array}}{F}$$

The Curry–Howard correspondence II

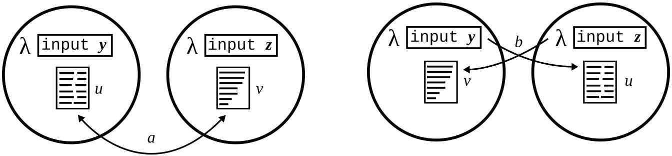
$$\dfrac{\begin{matrix}\vdots \\ t : A \supset B\end{matrix} \qquad \begin{matrix}\vdots \\ u : A\end{matrix}}{tu : B} \qquad\qquad \dfrac{\begin{matrix}[x : A] \\ \vdots \\ t : B\end{matrix}}{\lambda x\, t : A \supset B}$$

$$\dfrac{\begin{matrix}\vdots \\ \dfrac{s : A}{cs : B} \\ \vdots \\ u : F\end{matrix} \qquad \begin{matrix}\vdots \\ \dfrac{t : B}{ct : A} \\ \vdots \\ v : F\end{matrix}}{u \parallel_c v : F}$$

A parallel λ-calculus

[Aschieri, Ciabattoni and Genco. Submitted.]

- Normalisation procedure
- Subformula property
- Meaningful computational reductions
  (*e.g.*, in terms of code optimisation)



Future work

Find **computational interpretations** of logics
that are formalised by hypersequent calculi

Future work

Find **computational interpretations** of logics
that are formalised by hypersequent calculi

All axiomatisable propositional intermediate logics are defined
by ***canonical formulae*** [Chagrov and Zakharyaschev, 1997]
which are **just outside the scope of hypersequents**

Future work

Find **computational interpretations** of logics
that are formalised by hypersequent calculi

All axiomatisable propositional intermediate logics are defined
by ***canonical formulae*** [Chagrov and Zakharyaschev, 1997]
which are **just outside the scope of hypersequents**

Three-level **systems of rules** seem a very promising option

Avron, A. (1987).
A constructive analysis of RM.
J. Symbolic Logic, 52(4):939–951.

Avron, A. (1991).
Hypersequents, logical consequence and intermediate logics for concurrency.
Ann. Math. Artif. Intell., 4(3):225–248.

Chagrov, A. and Zakharyaschev, M. (1997).
Modal Logic, volume 35 of *Oxford Logic Guides*.
Oxford University Press.

Ciabattoni, A., Galatos, N., and Terui, K. (2008).
From axioms to analytic rules in nonclassical logics.
In *LICS 2008*, pages 229–240.

Ciabattoni, A., Galatos, N., and Terui, K. (2012).
Algebraic proof theory for substructural logics: Cut-elimination and completions.
Ann. Pure Appl. Logic, 163(3):266–290.

Fitting, M. (2012).
Prefixed tableaus and nested sequents.
Ann. Pure Appl. Logic, 163(3):291–313.

Goré, R. and Ramanayake, R. (2012).
Labelled tree sequents, tree hypersequents and nested (deep) sequents.
In *Advances in Modal Logic*, pages 279–299.

Mints, G. E. (1968).
Some calculi of modal logic.
In *Logical and logical-mathematical calculus, part I*, volume 98 of *Trudy Mat. Inst. Steklov.*, pages 88–111.

Mints, G. E. (1992).
Selected papers in proof theory.
Bibliopolis.

Negri, S. (2014).
Proof analysis beyond geometric theories: from rule systems to systems of rules.
J. Logic Comput.

Pottinger, G. (1983).
Uniform, cut-free formulations of T, S4 and S5.
J. Symbolic Logic, 48(3):900.

Ramanayake, R. (2015).
Embedding the hypersequent calculus in the display calculus.
J. Logic Comput., 3:921–942.

Ramanayake, R. (2016).
Inducing syntactic cut-elimination for indexed nested sequents.
In *IJCAR 2016*, pages 416–432.

Wansing, H. (1998).
Translation of hypersequents into display sequents.
Log. J. IGPL, 6(5):719–733.