Structural multi-type sequent calculus for modal intuitionistic dependence logic

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- Henkin quantifiers (1961), Independence-friendly logic (Hintikka, Sandu 1989)
- First-order dependence logic (Väänänen 2007)

- Modal dependence logic (Väänänen 2008):

$$\text{Modal Logic} \quad + \quad =(\vec{p}, q)$$

- Modal intuitionistic dependence logic (**MID**):

$$\phi ::= p \mid \bot \mid =(p_1, \ldots, p_n, q) \mid \phi \wedge \phi \mid \phi \to \phi \mid \phi \vee \phi \mid \Box\phi \mid \Diamond\phi$$

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SYNTAX:

- Modal intuitionistic dependence logic (**MID**):

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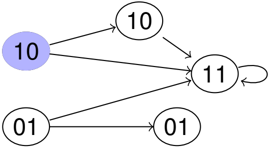
Team Semantics (Hodges 1997)

- $\mathfrak{M} = (W, R, V)$
- A team $X \subseteq W$



$\mathfrak{M}, w \models =\!(\vec{p}, q)$

(Empty Team Property) $\mathfrak{M}, \emptyset \models \phi$.
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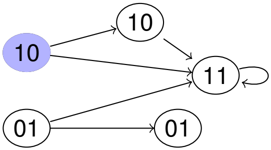

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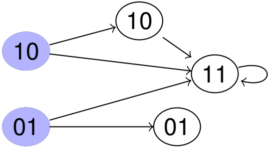

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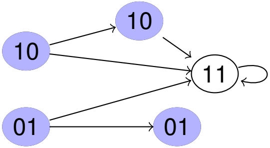

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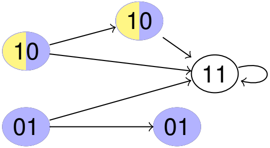

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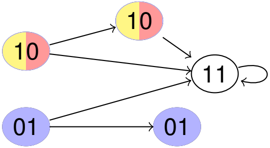

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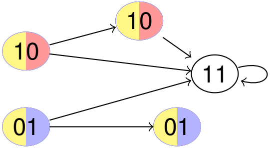

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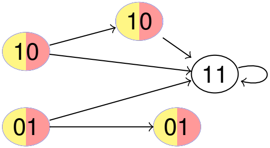

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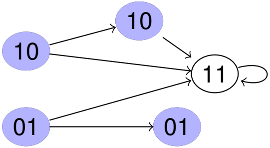

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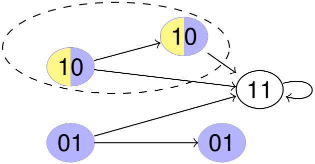

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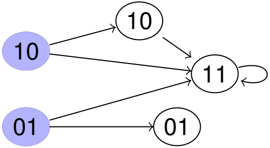

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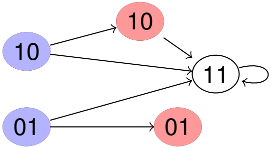

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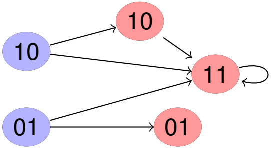

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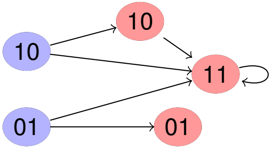

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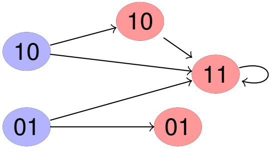

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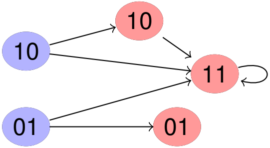

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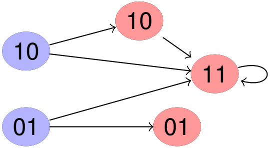

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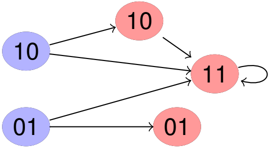

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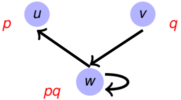
A (modal) Kripke model $\mathfrak{M} = (W, R, V)$



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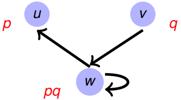
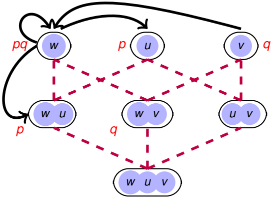


$\downarrow$

The *powerset model* $\mathfrak{M}^\circ = (W^\circ, \supseteq, R^\circ, V^\circ)$ *induced by* $\mathfrak{M}$:

- $W^\circ = \wp(W) \setminus \{\emptyset\}$
- $XR^\circ Y$ iff XRY
- $X \in V^\circ(p)$ iff $X \subseteq V(p)$



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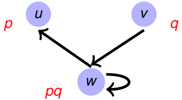

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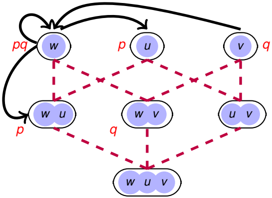

Fact:

- $\mathfrak{M}^\circ$ is a bi-relation Kripke model of Fischer Servi's intuitionistic modal logic, and $\mathfrak{M}, X \models \phi \iff \mathfrak{M}^\circ, X \Vdash \phi$.

$\mathfrak{M}, X \models \phi \to \psi$ iff for all $Y \subseteq X$, if $\mathfrak{M}, Y \models \phi$ then $\mathfrak{M}, Y \models \psi$
$\mathfrak{M}, X \models \phi \vee \psi$ iff $\mathfrak{M}, X \models \phi$ or $\mathfrak{M}, X \models \psi$

- $(W^\circ, \supseteq)$ is a Medvedev frame and a frame of Krisel-Putnam logic
- V° is a negative valuation: $\mathfrak{M}^\circ, X \Vdash \neg\neg p \iff \mathfrak{M}^\circ, X \Vdash p$

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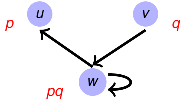

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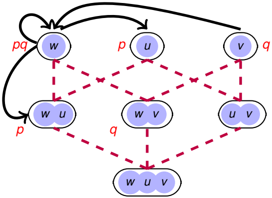

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$\mathfrak{M}, X \models \Box(\phi \vee \psi) \to \Box\phi \vee \Box\psi$

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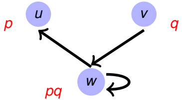
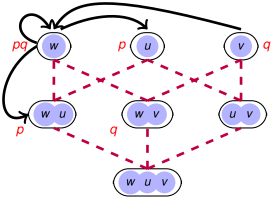
- V° is a negative valuation: $\mathfrak{M}^\circ, X \Vdash \neg\neg p \iff \mathfrak{M}^\circ, X \Vdash p$

A (modal) Kripke model $\mathfrak{M} = (W, R, V)$



The *powerset model* $\mathfrak{M}^\circ = (W^\circ, \supseteq, R^\circ, V^\circ)$ *induced by* $\mathfrak{M}$:

- $W^\circ = \wp(W) \setminus \{\emptyset\}$
- $XR^\circ Y$ iff XRY
- $X \in V^\circ(p)$ iff $X \subseteq V(p)$



Fact:

- $\mathfrak{M}^\circ$ is a bi-relation Kripke model of Fischer Servi's intuitionistic modal logic, and $\mathfrak{M}, X \models \phi \iff \mathfrak{M}^\circ, X \Vdash \phi$.

$\mathfrak{M}, X \models \phi \to \psi$ iff for all $Y \subseteq X$, if $\mathfrak{M}, Y \models \phi$ then $\mathfrak{M}, Y \models \psi$
$\mathfrak{M}, X \models \phi \vee \psi$ iff $\mathfrak{M}, X \models \phi$ or $\mathfrak{M}, X \models \psi$

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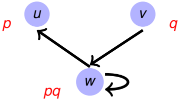

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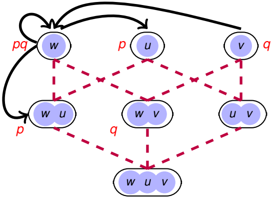

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A Hilbert style system of **MID**

Axioms:

1. axioms of Fischer Servi's intuitionistic modal logic **IK**
   1. all axioms of **IPC**
   2. $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$
   3. $\Box(\phi \to \psi) \to (\Diamond\phi \to \Diamond\psi)$
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Inquisitive Logic (Ciardelli and Roelofsen 2009)

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A multi-type Sequent Calculus for Inquisitive Logic
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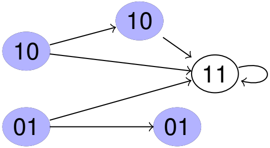

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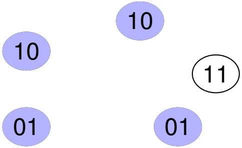

Flat Type

General Type

Fix a set V of propositional variables.

Flat Type

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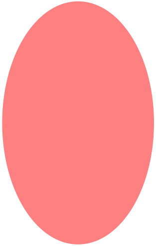
$$\mathbb{B} = (\wp(2^V), \cap, \cup, (\cdot)^c, \emptyset, 2^V)$$



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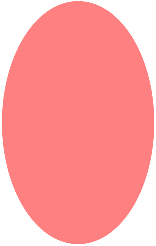

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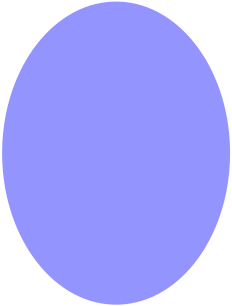

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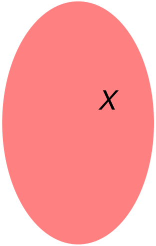

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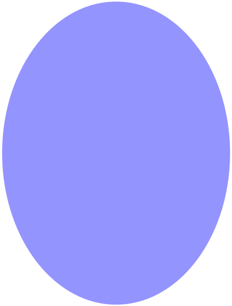

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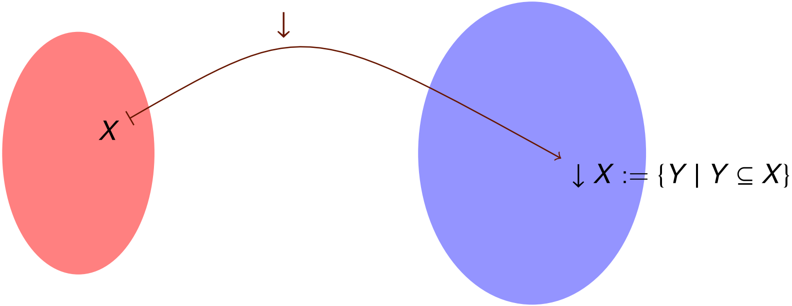

$\downarrow$

X

$\downarrow X := \{Y \mid Y \subseteq X\}$

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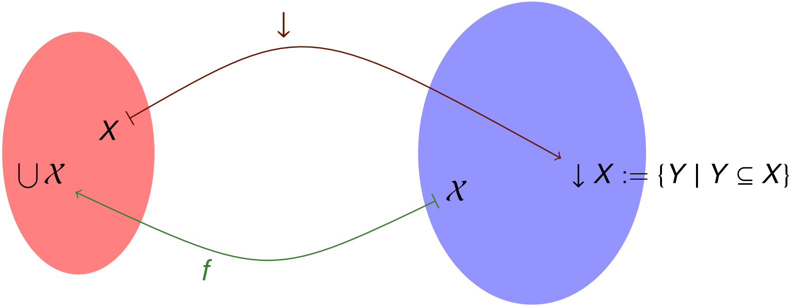

$$f \dashv \downarrow$$

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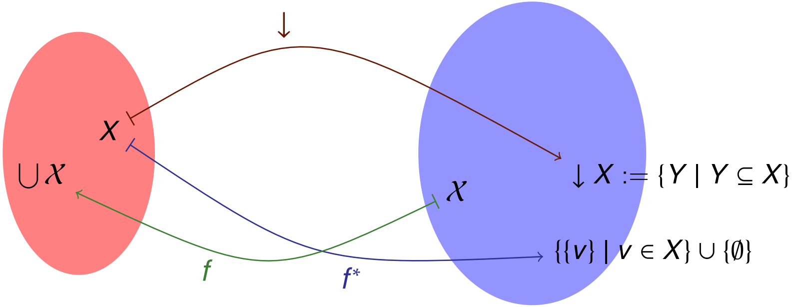

$\downarrow$

X

$\bigcup \mathcal{X}$

$\mathcal{X}$

$\downarrow X := \{Y \mid Y \subseteq X\}$

$\{\{v\} \mid v \in X\} \cup \{\emptyset\}$

f

f^*

$f^* \dashv f \dashv \downarrow$

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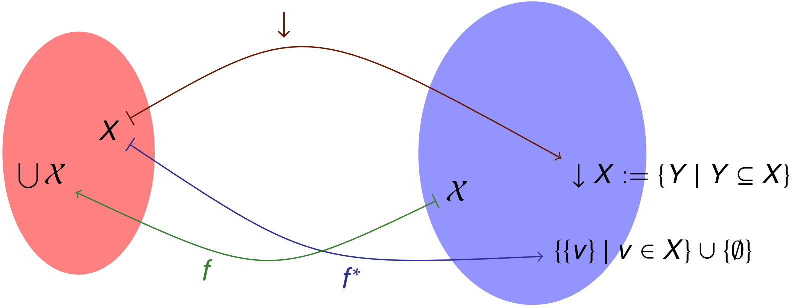
General Type

$$\mathbb{A} = (\wp^{\downarrow}(2^V), \cap, \cup, \Rightarrow, \emptyset, \wp(2^V))$$



$$f^* \dashv f \dashv \downarrow$$

Multi-type inquisitive logic:

$$\text{Flat} \ni \alpha ::= p \mid 0 \mid \alpha \sqcap \alpha \mid \alpha \rightharpoonup \alpha$$

$$\text{General} \ni A ::= \downarrow\alpha \mid A \wedge A \mid A \vee A \mid A \rightarrow A$$

Multi-type modal intuitionistic dependence logic:

$$\textcolor{red}{\text{Flat}} \ni \alpha ::= p \mid 0 \mid \alpha \sqcap \alpha \mid \alpha \rightarrowtriangle \alpha \mid \Diamonddot\alpha \mid \boxdot\alpha$$

$$\textcolor{blue}{\text{General}} \ni A ::= \downarrow\alpha \mid A \wedge A \mid A \vee A \mid A \to A \mid \Diamond A \mid \Box A$$

Multi-type modal intuitionistic dependence logic:

$$\text{Flat} \ni \alpha ::= \ p \mid 0 \mid \alpha \sqcap \alpha \mid \alpha \rightarrowtriangle \alpha \mid \Diamonddot\alpha \mid \boxdot\alpha$$

$$\text{General} \ni A ::= \ \downarrow\alpha \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid \Diamond A \mid \Box A$$

Axioms:

1. axiom schemata of **K** for Flat-formulas:
    - (1) axiom schemata of **CPC**
    - (2) $\boxdot(\alpha \rightarrowtriangle \beta) \rightarrowtriangle (\boxdot\alpha \rightarrowtriangle \boxdot\beta)$
    - (3) $\Diamonddot\alpha \leftrightarrowtriangle \sim\boxdot\sim\alpha$
2. $\neg\neg\downarrow\alpha \rightarrow \downarrow\alpha$
3. $(\downarrow\alpha \rightarrow A \vee B) \rightarrow (\downarrow\alpha \rightarrow A) \vee (\downarrow\alpha \rightarrow B)$
4. axiom schemata of Fischer Servi's **IK** for General-formulas:
    - (1) axiom schemata of **IPC**
    - (2) $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
    - (3) $\Diamond(A \rightarrow B) \rightarrow (\Box A \rightarrow \Diamond B)$
    - (4) $\neg\Diamond\downarrow 0$
    - (5) $\Diamond(A \vee B) \rightarrow (\Diamond A \vee \Diamond B)$
    - (6) $(\Diamond A \rightarrow \Box B) \rightarrow \Box(A \rightarrow B)$
5. $\Box(A \vee B) \rightarrow (\Box A \vee \Box B)$
6. $\neg\Diamond\neg\downarrow\alpha \rightarrow \Box\downarrow\alpha$

Rules: Modus Ponens and Necessitation for formulas of both types

Multi-type modal intuitionistic dependence logic:

$$\text{Flat} \ni \alpha ::= p \mid 0 \mid \alpha \sqcap \alpha \mid \alpha \rightarrowtriangle \alpha \mid \diamonddot \alpha \mid \boxdot \alpha$$

$$\text{General} \ni A ::= {\downarrow}\alpha \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid \Diamond A \mid \Box A$$

Axioms:

1. axiom schemata of **K** for Flat-formulas:
   - (1) axiom schemata of **CPC**
   - (2) $\boxdot(\alpha \rightarrowtriangle \beta) \rightarrowtriangle (\boxdot\alpha \rightarrowtriangle \boxdot\beta)$
   - (3) $\diamonddot\alpha \leftrightarrowtriangle \sim \boxdot \sim\alpha$
2. $\neg\neg{\downarrow}\alpha \rightarrow {\downarrow}\alpha$
3. $({\downarrow}\alpha \rightarrow A \vee B) \rightarrow ({\downarrow}\alpha \rightarrow A) \vee ({\downarrow}\alpha \rightarrow B)$
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   - (3) $\Diamond(A \rightarrow B) \rightarrow (\Box A \rightarrow \Diamond B)$
   - (4) $\neg\Diamond{\downarrow}0$
   - (5) $\Diamond(A \vee B) \rightarrow (\Diamond A \vee \Diamond B)$
   - (6) $(\Diamond A \rightarrow \Box B) \rightarrow \Box(A \rightarrow B)$
5. $\Box(A \vee B) \rightarrow (\Box A \vee \Box B)$
6. $\neg\Diamond\neg{\downarrow}\alpha \rightarrow \Box{\downarrow}\alpha$

Rules: Modus Ponens and Necessitation for formulas of both types

Multi-type modal intuitionistic dependence logic:

$$\text{Flat} \ni \alpha ::= p \mid 0 \mid \alpha \sqcap \alpha \mid \alpha \rightarrowtriangle \alpha \mid \Diamonddot\alpha \mid \boxdot\alpha$$

$$\text{General} \ni A ::= {\downarrow}\alpha \mid A \wedge A \mid A \vee A \mid A \to A \mid \Diamond A \mid \Box A$$

Axioms:

1. axiom schemata of **K** for Flat-formulas:
   (1) axiom schemata of **CPC**
   (2) $\boxdot(\alpha \rightarrowtriangle \beta) \rightarrowtriangle (\boxdot\alpha \rightarrowtriangle \boxdot\beta)$
   (3) $\Diamonddot\alpha \leftrightarrowtriangle {\sim}\boxdot{\sim}\alpha$
2. $\neg\neg{\downarrow}\alpha \to {\downarrow}\alpha$
3. $({\downarrow}\alpha \to A \vee B) \to ({\downarrow}\alpha \to A) \vee ({\downarrow}\alpha \to B)$
4. axiom schemata of Fischer Servi's **IK** for General-formulas:
   (1) axiom schemata of **IPC**
   (2) $\Box(A \to B) \to (\Box A \to \Box B)$
   (3) $\Diamond(A \to B) \to (\Box A \to \Diamond B)$
   (4) $\neg\Diamond{\downarrow}0$
   (5) $\Diamond(A \vee B) \to (\Diamond A \vee \Diamond B)$
   (6) $(\Diamond A \to \Box B) \to \Box(A \to B)$
5. $\Box(A \vee B) \to (\Box A \vee \Box B)$
6. $\neg\Diamond\neg{\downarrow}\alpha \to \Box{\downarrow}\alpha$

Rules: Modus Ponens and Necessitation for formulas of both types

Display Calculi

- Natural generalization of Gentzen's sequent calculi;
- sequents $X \vdash Y$, where X and Y are **structures**:
  - built by **structural connectives** (e.g. Gentzen's comma)
  - **binary trees** (not sequences)
- **Display property**: **adjunction** at the structural level
- **Canonical proof of cut elimination**

Theorem (Display Property)

Each substructure in a display-sequent is 'displayable' in precedent or, exclusively, succedent position.

Display Postulates

$$\begin{array}{c} X \,;\, Y \vdash Z \\ \hline\hline Y \vdash X > Z \end{array} \qquad \begin{array}{c} Z \vdash Y \,;\, X \\ \hline\hline Y > Z \vdash X \end{array}$$

A structure in display

$$\begin{array}{c} Y \vdash X > Z \\ \hline X \,;\, Y \vdash Z \\ \hline Y \,;\, X \vdash Z \\ \hline X \vdash Y > Z \end{array}$$

Rules

Operational

$$\frac{A ; B \vdash X}{A \wedge B \vdash X} \qquad \frac{X \vdash A \qquad Y \vdash B}{X ; Y \vdash A \wedge B}$$

$$\frac{X \vdash A \qquad B \vdash Y}{A \rightarrow B \vdash X > Y} \qquad \frac{X \vdash A > B}{X \vdash A \rightarrow B}$$

Structural

$$Gri_L \; \begin{array}{c} (X > Y); Z \vdash W \\ \hline\hline X > (Y; Z) \vdash W \end{array} \qquad \begin{array}{c} W \vdash (X > Y); Z \\ \hline\hline W \vdash X > (Y; Z) \end{array} \; Gri_R$$

$$FS_L \; \frac{\circ X > \circ Y \vdash Z}{\circ(X > Y) \vdash Z} \qquad \frac{X \vdash \circ Y > \circ Z}{X \vdash \circ(Y > Z)} \; FS_R$$

The excluded middle is derivable using (classically valid) *Grishin's rule*:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A ; \mathrm{I} \vdash A}}{A ; \mathrm{I} \vdash \bot ; A}}{\mathrm{I} \vdash A > (\bot ; A)}}{\mathrm{I} \vdash (A > \bot) ; A}\,\mathit{Gri}}{\mathrm{I} \vdash A ; (A > \bot)}}{A > \mathrm{I} \vdash A > \bot}}{A > \mathrm{I} \vdash A \to \bot}}{A > \mathrm{I} \vdash \neg A}}{\mathrm{I} \vdash A ; \neg A}}{\mathrm{I} \vdash A \vee \neg A}$$

The intutionistic modal axioms are derivable using *Fisher Servi's rules*:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\circ A \vdash \Diamond A} \qquad \dfrac{B \vdash B}{\Box B \vdash \circ B}}{\Diamond A \rightarrow \Box B \vdash \circ A > \circ B}}{\Diamond A \rightarrow \Box B \vdash \circ (A > B)}\ FS}{\bullet(\Diamond A \rightarrow \Box B) \vdash A > B}}{\bullet(\Diamond A \rightarrow \Box B) \vdash A \rightarrow B}}{\Diamond A \rightarrow \Box B \vdash \circ(A \rightarrow B)}}{\Diamond A \rightarrow \Box B \vdash \Box(A \rightarrow B)}}{(\Diamond A \rightarrow \Box B)\,;\mathrm{I} \vdash \Box(A \rightarrow B)}}{\mathrm{I} \vdash (\Diamond A \rightarrow \Box B) > \Box(A \rightarrow B)}}{\mathrm{I} \vdash (\Diamond A \rightarrow \Box B) \rightarrow \Box(A \rightarrow B)}
$$

Canonical cut elimination, 1/2

Definition

A sequent $x \vdash y$ is *type-uniform* if x and y are of the same type.

A (cut) rule is *strongly type-uniform* if its premises and conclusion are of the same type.

Theorem (Canonical cut elimination)

If a calculus satisfies the properties below, then it enjoys cut elimination.

Canonical cut elimination, 2/2

1. structures can disappear, formulas are **forever**;
2. **tree-traceable** formula-occurrences, via suitably defined congruence:
    - same shape, same position, same type, non-proliferation;
3. **principal = displayed**
4. rules are closed under **uniform substitution** of congruent parameters within each type;
5. **reduction strategy** exists when cut formulas are both principal.

Specific to multi-type setting:

6. **type-uniformity** of derivable sequents;
7. **strongly uniform cuts** in each/some type(s).

Structural and operational languages

Flat

$\alpha ::= p \mid 0 \mid \alpha \sqcap \alpha \mid \alpha \mathrel{-\!\!\triangleright} \alpha \mid \diamonddot\alpha \mid \boxdot\alpha$

$\Gamma ::= \alpha \mid \Phi \mid \Gamma , \Gamma \mid \Gamma \sqsupset \Gamma \mid \mathrm{F}X \mid \odot\Gamma \mid \circledbullet\Gamma$

General

$A ::= {\downarrow}\alpha \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid \Diamond A \mid \Box A$

$X ::= A \mid {\Downarrow}\Gamma \mid \mathrm{F}^{*}\Gamma \mid X ; X \mid X > X \mid \circ X \mid \bullet X$

Structural and operational languages

Flat	General
$\alpha ::= p \mid 0 \mid \alpha \sqcap \alpha \mid \alpha \rightarrowtriangle \alpha \mid ⟐\alpha \mid \boxdot\alpha$	$A ::= \downarrow\alpha \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid \Diamond A \mid \Box A$
$\Gamma ::= \alpha \mid \Phi \mid \Gamma \,,\, \Gamma \mid \Gamma \sqsupset \Gamma \mid \mathrm{F}X \mid \odot\Gamma \mid ◉\Gamma$	$X ::= A \mid \Downarrow\Gamma \mid \mathrm{F}^*\Gamma \mid X \,;\, X \mid X > X \mid \circ X \mid \bullet X$

Interpretation of structural connectives

Flat connectives:

Structural symbols	Φ		$,$		$\sqsupset$		$\odot$		$◉$	
Operational symbols	(1)	0	$\sqcap$	$(\sqcup)$	$(\mapsto)$	$\rightarrowtriangle$	$⟐$	$\boxdot$	$(◈)$	$(▣)$

General connectives:

Structural symbols	$;$		$>$		$\circ$		$\bullet$	
Operational symbols	$\wedge$	$\vee$	$(\rightarrowtail)$	$\rightarrow$	$\Diamond$	$\Box$	$(\blacklozenge)$	$(\blacksquare)$

Multi-type connectives:

Structural symbols	F^*		F		$\Downarrow$	
Operational symbols	(f^*)		(f)	(f)	$\downarrow$	$\downarrow$

Structural and operational languages

Flat	General
$\alpha ::= p \mid 0 \mid \alpha \sqcap \alpha \mid \alpha \rightarrowtriangle \alpha \mid \Diamonddot\alpha \mid \boxdot\alpha$	$A ::= \downarrow\alpha \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid \Diamond A \mid \Box A$
$\Gamma ::= \alpha \mid \Phi \mid \Gamma\,,\Gamma \mid \Gamma \sqsupset \Gamma \mid \mathrm{F}X \mid \odot\Gamma \mid \circledbullet\Gamma$	$X ::= A \mid \Downarrow\Gamma \mid \mathrm{F}^*\Gamma \mid X \,;X \mid X > X \mid \circ X \mid \bullet X$

Interpretation of structural connectives

Flat connectives:

Structural symbols	Φ		,		⊐		⊙		⦿	
Operational symbols	(1)	0	⊓	(⊔)	(↦)	⇾	⟐	⊡	(◆)	(■)

General connectives:

Structural symbols	;		>		∘		•	
Operational symbols	∧	∨	(↣)	→	◇	□	(◆)	(■)

Multi-type connectives:

$(f^* \dashv f \dashv \ \downarrow)$

Structural symbols	F*		F		⇓	
Operational symbols	(f*)		(f)	(f)	↓	↓

Cut rules

$$\frac{\Gamma \vdash \alpha \qquad (\Sigma \vdash \Delta)[\alpha]^{pre}}{(\Sigma \vdash \Delta)[\Gamma/\alpha]^{pre}}\ \mathit{Cut} \qquad\qquad \frac{X \vdash A \qquad A \vdash Y}{X \vdash Y}\ \mathit{Cut}$$

Structural rules (propositional)

Flat type:

$$\mathit{Id}\ \frac{}{p \vdash p} \qquad\qquad \frac{\Pi \vdash \Gamma \sqsupset (\Delta \,, \Sigma)}{\Pi \vdash (\Gamma \sqsupset \Delta) \,, \Sigma}\ \mathrm{CG}$$

Interaction between the two types:

$$\frac{\Gamma \vdash \Delta}{\mathrm{F}^*\Gamma \vdash \Downarrow\Delta}\ \text{bal} \qquad \frac{\Gamma \vdash \Delta}{\Downarrow\Gamma \vdash \Downarrow\Delta}\ \text{d mon} \qquad \frac{X \vdash Y}{\mathrm{F}X \vdash \mathrm{F}Y}\ \text{f mon}$$

$$\begin{array}{c} \mathrm{F}^*\Gamma \vdash \Delta \\ \hline\hline \Gamma \vdash \mathrm{F}\Delta \end{array}\ \text{f adj} \qquad \begin{array}{c} \mathrm{F}X \vdash \Gamma \\ \hline\hline X \vdash \Downarrow\Gamma \end{array}\ \text{d adj} \qquad \frac{\Downarrow\mathrm{F}X \vdash Y}{X \vdash Y}\ \text{d-f elim}$$

$$\begin{array}{c} X \vdash \Downarrow(\Gamma \sqsupset \Delta) \\ \hline\hline X \vdash \Downarrow\Gamma > \Downarrow\Delta \end{array}\ \text{d dis} \qquad \begin{array}{c} \mathrm{F}X \,, \mathrm{F}Y \vdash Z \\ \hline\hline \mathrm{F}(X \,; Y) \vdash Z \end{array}\ \text{f dis}$$

$$\frac{X \vdash \Downarrow\Gamma > (Y \,; Z) \qquad X \vdash \Downarrow\Gamma > (Y \,; Z)}{X \vdash (\Downarrow\Gamma > Y) \,; (\Downarrow\Gamma > Z)}\ \mathrm{KP}$$

Cut rules

$$\frac{\Gamma \vdash \alpha \qquad (\Sigma \vdash \Delta)[\alpha]^{pre}}{(\Sigma \vdash \Delta)[\Gamma/\alpha]^{pre}}\ \mathit{Cut} \qquad\qquad \frac{X \vdash A \qquad A \vdash Y}{X \vdash Y}\ \mathit{Cut}$$

Structural rules (propositional)

Flat type:

$$\mathit{Id}\ \frac{}{p \vdash p} \qquad\qquad \frac{\Pi \vdash \Gamma \sqsupset (\Delta\,,\Sigma)}{\Pi \vdash (\Gamma \sqsupset \Delta)\,,\Sigma}\ \mathrm{CG}$$

Interaction between the two types:

$$\frac{\Gamma \vdash \Delta}{\mathrm{F}^{*}\Gamma \vdash \Downarrow\Delta}\ \text{bal} \qquad \frac{\Gamma \vdash \Delta}{\Downarrow\Gamma \vdash \Downarrow\Delta}\ \text{d mon} \qquad \frac{X \vdash Y}{\mathrm{F}X \vdash \mathrm{F}Y}\ \text{f mon}$$

$$\begin{array}{c} \mathrm{F}^{*}\Gamma \vdash \Delta \\ \hline\hline \Gamma \vdash \mathrm{F}\Delta \end{array}\ \text{f adj} \qquad \begin{array}{c} \mathrm{F}X \vdash \Gamma \\ \hline\hline X \vdash \Downarrow\Gamma \end{array}\ \text{d adj} \qquad \frac{\Downarrow\mathrm{F}X \vdash Y}{X \vdash Y}\ \text{d-f elim}$$

$$\begin{array}{c} X \vdash \Downarrow(\Gamma \sqsupset \Delta) \\ \hline\hline X \vdash \Downarrow\Gamma > \Downarrow\Delta \end{array}\ \text{d dis} \qquad \begin{array}{c} \mathrm{F}X\,,\mathrm{F}Y \vdash Z \\ \hline\hline \mathrm{F}(X\,;Y) \vdash Z \end{array}\ \text{f dis}$$

$$\frac{X \vdash \Downarrow\Gamma > (Y\,;Z) \qquad X \vdash \Downarrow\Gamma > (Y\,;Z)}{X \vdash (\Downarrow\Gamma > Y)\,;(\Downarrow\Gamma > Z)}\ \text{KP}$$

Cut rules

$$\frac{\Gamma \vdash \alpha \qquad (\Sigma \vdash \Delta)[\alpha]^{pre}}{(\Sigma \vdash \Delta)[\Gamma / \alpha]^{pre}}\ \mathit{Cut} \qquad\qquad \frac{X \vdash A \qquad A \vdash Y}{X \vdash Y}\ \mathit{Cut}$$

Structural rules (propositional)

Flat type:

$$\mathit{Id}\ \frac{}{p \vdash p} \qquad\qquad \frac{\Pi \vdash \Gamma \sqsupset (\Delta \,,\Sigma)}{\Pi \vdash (\Gamma \sqsupset \Delta)\,,\Sigma}\ \mathrm{CG}$$

Interaction between the two types:

$$\frac{\Gamma \vdash \Delta}{\mathrm{F}^{*}\Gamma \vdash \Downarrow\Delta}\ \text{bal} \qquad \frac{\Gamma \vdash \Delta}{\Downarrow\Gamma \vdash \Downarrow\Delta}\ \text{d mon} \qquad \frac{X \vdash Y}{\mathrm{F}X \vdash \mathrm{F}Y}\ \text{f mon}$$

$$\begin{array}{c} \mathrm{F}^{*}\Gamma \vdash \Delta \\ \hline\hline \Gamma \vdash \mathrm{F}\Delta \end{array}\ \text{f adj} \qquad \begin{array}{c} \mathrm{F}X \vdash \Gamma \\ \hline\hline X \vdash \Downarrow\Gamma \end{array}\ \text{d adj} \qquad \frac{\Downarrow\mathrm{F}X \vdash Y}{X \vdash Y}\ \text{d-f elim}$$

$$\begin{array}{c} X \vdash \Downarrow(\Gamma \sqsupset \Delta) \\ \hline\hline X \vdash \Downarrow\Gamma > \Downarrow\Delta \end{array}\ \text{d dis} \qquad \begin{array}{c} \mathrm{F}X\,,\mathrm{F}Y \vdash Z \\ \hline\hline \mathrm{F}(X\,;\,Y) \vdash Z \end{array}\ \text{f dis}$$

$$\frac{X \vdash \Downarrow\Gamma > (Y\,;Z) \qquad X \vdash \Downarrow\Gamma > (Y\,;Z)}{X \vdash (\Downarrow\Gamma > Y)\,;(\Downarrow\Gamma > Z)}\ \mathrm{KP}$$

Structural rules (for modalities)

Structural rules specific to the Gen type:

$$\frac{X \vdash \bullet(Y\,;Z)}{X \vdash \bullet Y\,;\bullet Z}\ \text{dis}$$

Structural rules governing the interaction between $\Downarrow$ and modalities:

$$\text{swap}\ \begin{array}{c} \ast\Downarrow\Gamma \vdash X \\ \hline\hline \Downarrow\circledast\Gamma \vdash X \end{array} \qquad\qquad \begin{array}{c} X \vdash \ast\Downarrow\Gamma \\ \hline\hline X \vdash \Downarrow\circledast\Gamma \end{array}\ \text{swap}$$

Structural rules common to both types: Let $\circledast \in \{\odot, \boldsymbol{\odot}\}$ and $\ast \in \{\circ, \bullet\}$.

$$\text{Flat adj}\ \begin{array}{c} \odot\Gamma \vdash \Delta \\ \hline\hline \Gamma \vdash \boldsymbol{\odot}\Delta \end{array} \qquad \begin{array}{c} \Gamma \vdash \odot\Delta \\ \hline\hline \boldsymbol{\odot}\Gamma \vdash \Delta \end{array}\ \text{Flat adj} \quad \text{Gen adj}\ \begin{array}{c} \circ X \vdash Y \\ \hline\hline X \vdash \bullet Y \end{array} \qquad \begin{array}{c} X \vdash \circ Y \\ \hline\hline \bullet X \vdash Y \end{array}\ \text{Gen adj}$$

$$\text{nec}\ \frac{\Phi \vdash \Gamma}{\circledast\Phi \vdash \Gamma} \qquad \frac{\Gamma \vdash \Phi}{\Gamma \vdash \circledast\Phi}\ \text{nec} \qquad\qquad \text{nec}\ \frac{\mathrm{I} \vdash X}{\ast\mathrm{I} \vdash X} \qquad \frac{X \vdash \mathrm{I}}{X \vdash \ast\mathrm{I}}\ \text{nec}$$

$$\text{FS}\ \frac{\circledast\Gamma \sqsupset \circledast\Delta \vdash \Sigma}{\circledast(\Gamma \sqsupset \Delta) \vdash \Sigma} \qquad \frac{Y \vdash \circledast X \sqsupset \circledast Z}{Y \vdash \circledast(X \sqsupset Z)}\ \text{FS} \quad \text{FS}\ \frac{\ast Y > \ast Z \vdash X}{\ast(Y > Z) \vdash X} \qquad \frac{Y \vdash \ast X > \ast Z}{Y \vdash \ast(X > Z)}\ \text{FS}$$

$$\text{mon}\ \frac{\circledast\Gamma\,,\circledast\Delta \vdash \Sigma}{\circledast(\Gamma\,,\Delta) \vdash \Sigma} \qquad \frac{\Gamma \vdash \circledast\Delta\,,\circledast\Sigma}{\Gamma \vdash \circledast(\Delta\,,\Sigma)}\ \text{mon}$$

Structural rules (for modalities)

Structural rules specific to the Gen type: $(\Box(\phi \vee \psi) \rightarrow \Box\phi \vee \Box\psi)$

$$\dfrac{X \vdash \bullet(Y \mathbin{;} Z)}{X \vdash \bullet Y \mathbin{;} \bullet Z}\ \mathrm{dis}$$

Structural rules governing the interaction between $\Downarrow$ and modalities:

$$\mathrm{swap}\ \begin{array}{c} \ast \Downarrow \Gamma \vdash X \\ \hline\hline \Downarrow \circledast \Gamma \vdash X \end{array} \qquad \begin{array}{c} X \vdash \ast \Downarrow \Gamma \\ \hline\hline X \vdash \Downarrow \circledast \Gamma \end{array}\ \mathrm{swap}$$

Structural rules common to both types: Let $\circledast \in \{\odot, \circledbullet\}$ and $\ast \in \{\circ, \bullet\}$.

$$\text{Flat adj}\ \begin{array}{c} \odot\Gamma \vdash \Delta \\ \hline\hline \Gamma \vdash \circledbullet\Delta \end{array} \quad \begin{array}{c} \Gamma \vdash \odot\Delta \\ \hline\hline \circledbullet\Gamma \vdash \Delta \end{array}\ \text{Flat adj} \quad \text{Gen adj}\ \begin{array}{c} \circ X \vdash Y \\ \hline\hline X \vdash \bullet Y \end{array} \quad \begin{array}{c} X \vdash \circ Y \\ \hline\hline \bullet X \vdash Y \end{array}\ \text{Gen adj}$$

$$\mathrm{nec}\ \dfrac{\Phi \vdash \Gamma}{\circledast\Phi \vdash \Gamma} \quad \dfrac{\Gamma \vdash \Phi}{\Gamma \vdash \circledast\Phi}\ \mathrm{nec} \qquad \mathrm{nec}\ \dfrac{\mathrm{I} \vdash X}{\ast\mathrm{I} \vdash X} \quad \dfrac{X \vdash \mathrm{I}}{X \vdash \ast\mathrm{I}}\ \mathrm{nec}$$

$$\mathrm{FS}\ \dfrac{\circledast\Gamma \sqsupset \circledast\Delta \vdash \Sigma}{\circledast(\Gamma \sqsupset \Delta) \vdash \Sigma} \quad \dfrac{Y \vdash \circledast X \sqsupset \circledast Z}{Y \vdash \circledast(X \sqsupset Z)}\ \mathrm{FS} \quad \mathrm{FS}\ \dfrac{\ast Y > \ast Z \vdash X}{\ast(Y > Z) \vdash X} \quad \dfrac{Y \vdash \ast X > \ast Z}{Y \vdash \ast(X > Z)}\ \mathrm{FS}$$

$$\mathrm{mon}\ \dfrac{\circledast\Gamma \mathbin{,} \circledast\Delta \vdash \Sigma}{\circledast(\Gamma \mathbin{,} \Delta) \vdash \Sigma} \qquad \dfrac{\Gamma \vdash \circledast\Delta \mathbin{,} \circledast\Sigma}{\Gamma \vdash \circledast(\Delta \mathbin{,} \Sigma)}\ \mathrm{mon}$$

$$
\begin{prooftree}
\AxiomC{$\alpha \vdash \alpha$}
\UnaryInfC{$\alpha \vdash 0 , \alpha$}
\UnaryInfC{$\alpha , \Phi \vdash 0 , \alpha$}
\UnaryInfC{$\Phi \vdash {\color{blue}\alpha \sqsupset (0 , \alpha)}$}
\RightLabel{\color{blue}CG}
\UnaryInfC{$\Phi \vdash {\color{blue}(\alpha \sqsupset 0) , \alpha}$}
\UnaryInfC{$\Phi \vdash \alpha , (\alpha \sqsupset 0)$}
\UnaryInfC{$\alpha \sqsupset \Phi \vdash \alpha \sqsupset 0$}
\UnaryInfC{$\Downarrow(\alpha \sqsupset \Phi) \vdash \Downarrow(\alpha \sqsupset 0)$}
\RightLabel{d dis}
\UnaryInfC{$\Downarrow(\alpha \sqsupset \Phi) \vdash \Downarrow\alpha > \Downarrow 0$}
\UnaryInfC{$\circ\Downarrow(\alpha \sqsupset \Phi) \vdash \Diamond(\Downarrow\alpha > \Downarrow 0)$}
\LeftLabel{swap}
\UnaryInfC{$\Downarrow \odot (\alpha \sqsupset \Phi) \vdash \Diamond(\Downarrow\alpha > \Downarrow 0)$}
\AxiomC{$0 \vdash \Phi$}
\RightLabel{d mon}
\UnaryInfC{$\Downarrow 0 \vdash \Downarrow\Phi$}
\UnaryInfC{$\downarrow 0 \vdash \Downarrow\Phi$}
\BinaryInfC{$\Diamond(\Downarrow\alpha > \Downarrow 0) \rightarrow \downarrow 0 \vdash {\color{blue}\Downarrow \odot (\alpha \sqsupset \Phi) > \Downarrow\Phi}$}
\RightLabel{\color{blue}d dis}
\UnaryInfC{$\Diamond(\Downarrow\alpha > \Downarrow 0) \rightarrow \downarrow 0 \vdash {\color{blue}\Downarrow(\odot(\alpha \sqsupset \Phi) \sqsupset \Phi)}$}
\dashedLine
\UnaryInfC{$\Diamond(\Downarrow\alpha > \Downarrow 0) \rightarrow \downarrow 0 \vdash \Downarrow \odot \alpha$}
\UnaryInfC{$\Diamond(\Downarrow\alpha > \Downarrow 0) \rightarrow \downarrow 0 \vdash \circ\downarrow\alpha$}
\UnaryInfC{$\neg\Diamond\neg\downarrow\alpha \vdash \Box\downarrow\alpha$}
\end{prooftree}
$$

The system is

- sound and complete,
- a proper multi-type calculus (for which a Belnap-style cut elimination metatheorem exists),
- and thus enjoys Belnap-style cut elimination.

Future work: a calculus for propositional dependence logic

(Multi-type) propositional dependence logic:

$$\text{Flat} \ni \alpha ::= p \mid 0 \mid \alpha \sqcap \alpha \mid \alpha \rightarrowtriangle \alpha$$

$$\text{General} \ni A ::= \downarrow\alpha \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid A \otimes A$$

Future work: a calculus for propositional dependence logic

(Multi-type) propositional dependence logic:

$$\text{Flat} \ni \alpha ::= p \mid 0 \mid \alpha \sqcap \alpha \mid \alpha \multimap \alpha$$

$$\text{General} \ni A ::= {\downarrow}\alpha \mid A \wedge A \mid A \vee A \mid A \to A \mid A \otimes A$$

- Axioms:
  - (A1) **CPC** axiom schemata for Flat-formulas
  - (A2) **IPC** axiom schemata for General-formulas
  - (A3) $({\downarrow}\alpha \to (A \vee B)) \to ({\downarrow}\alpha \to A) \vee ({\downarrow}\alpha \to B)$
  - (A4) $\neg\neg{\downarrow}\alpha \to {\downarrow}\alpha$
  - (A5) $\phi \otimes (\psi \vee \chi) \leftrightarrow (\phi \otimes \psi) \vee (\phi \otimes \chi)$
  - (A6) $(\neg \downarrow \alpha \to \downarrow \beta) \leftrightarrow (\downarrow \alpha \otimes \downarrow \beta)$
- Rule:

  Modus Ponens for formulas of both types