Syntax meets semantics in abstract algebraic logic

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Outline and references

1. General remarks on Abstract Algebraic Logic
2. Bridge theorems and transfer theorems
3. Two open problems
4. Ordering protoalgebraic logics

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Algebraic Logic: the study of algebra-based semantics

Logics $\mathscr{L} = \langle \boldsymbol{Fm}, \vdash_{\mathscr{L}} \rangle \quad \longmapsto$ **algebra-based** semantics , i.e.,

any kind of semantics where :

1) models are :

algebras $\boldsymbol{A}$ + additional structure

2) interpretations are : $h \colon \boldsymbol{Fm} \to \boldsymbol{A}$

additional structure: semantic: $1 \in A \ , \ F \subseteq A \ , \ \mathscr{C} \subseteq \mathscr{P}(A)$

or syntactic: $\boldsymbol{\tau}(x) \subseteq Fm \times Fm$

The syntax is hidden inside algebra-based semantics

How much similar is the semantics to the logic?

Which properties of the logic are shared by the semantics?

Are there properties of the logic that are always shared by the semantics?

Assume you work in a framework where there is a criterion or procedure:

$\mathscr{L}$	$\longmapsto$	K	.... the algebraic
an arbitrary logic (perhaps only of a certain kind)		a class of algebra-based models	counterpart of $\mathscr{L}$

Bridge Theorem

For every logic $\mathscr{L}$ (perhaps only of a certain kind),

$\mathscr{L}$ satisfies **P** $\iff$ K satisfies **Q**

P: a syntactic property of a logic

Q: a semantic property of a class of models

Logic  Algebra

A special kind of Bridge Theorem, when **Q** is essentially the same as **P**:

Transfer Theorem

For every logic $\mathscr{L}$ (of a certain kind),

$\mathscr{L}$ satisfies **P** $\iff$ K satisfies **P**

- **P** has to be interpreted (perhaps slightly differently) in both sides (e.g., “to be finitely axiomatizable” , “to be decidable” , etc.).
- Or not: when **P** is a property of a generalized matrix, directly:

$$\langle \boldsymbol{Fm}, \mathcal{T}h\mathscr{L} \rangle \text{ or } \langle \boldsymbol{Fm}, \vdash_{\mathscr{L}} \rangle \text{ satisfies } \mathbf{P}$$

$$\Longrightarrow \quad \langle \boldsymbol{A}, \mathcal{F}i_{\mathscr{L}}\boldsymbol{A} \rangle \text{ or } \langle \boldsymbol{A}, Fg^{\boldsymbol{A}}_{\mathscr{L}} \rangle \text{ satisfies } \mathbf{P}, \text{ for all } \boldsymbol{A}\ ?$$

- We say: “The property **P** transfers from $\mathscr{L}$ to K” (converse trivial)

Bridge theorems and transfer theorems

are

the ultimate justification of Abstract Algebraic Logic

(and a major driving force in the evolution of the field)

Theorem

Let $\mathscr{L}$ be a logic. The following conditions are equivalent:

(i) $\mathscr{L}$ is finitary.

(ii) The class $\mathrm{Mod}\mathscr{L}$ is closed under ultraproducts.

(iii) For every algebra $\boldsymbol{A}$, the closure operator $Fg^{\boldsymbol{A}}_{\mathscr{L}}$ is finitary.

(i)$\Leftrightarrow$(ii): Bridge

(i)$\Leftrightarrow$(iii): Transfer

Characterizing classes in the Leibniz hierarchy

Theorem (Blok, Pigozzi, Czelakowski, 1986,1992)

Let $\mathscr{L}$ be a logic. The following conditions are equivalent:

(i) $\mathscr{L}$ is protoalgebraic.

(ii) There is a set $\Delta(x,y)$ of formulas (in at most two variables) satisfying:

$$\vdash_{\mathscr{L}} \Delta(x,x) \tag{R_Δ}$$

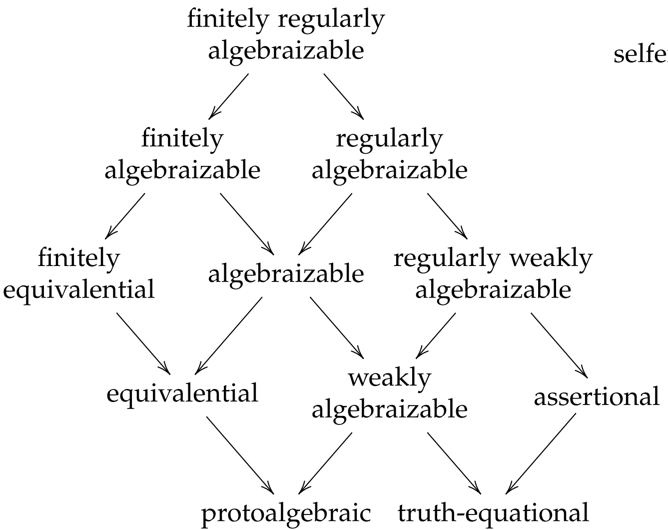
$$x, \Delta(x,y) \vdash_{\mathscr{L}} y \tag{MP_Δ}$$

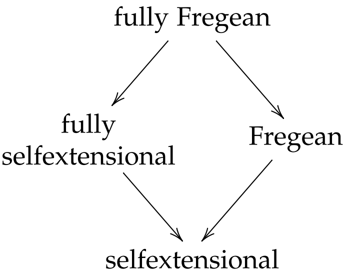
(iii) The class $\mathsf{Mod}^{*}\mathscr{L}$ is closed under subdirect products.

(iv) The Leibniz operator $\boldsymbol{\Omega}$ on the formula algebra is monotonic over the theories of $\mathscr{L}$.

(v) For every algebra $\boldsymbol{A}$, the Leibniz operator $\boldsymbol{\Omega}^{\boldsymbol{A}}$ is monotonic over the $\mathscr{L}$-filters of $\boldsymbol{A}$.

the Leibniz hierarchy





the Frege hierarchy

A TARSKI-style condition: the Inconsistency Lemma

(Reductio ad Absurdum for Intuitionistic Propositional Logic $\mathcal{I}\ell$)

For all $\Gamma \subseteq Fm$ and all $\alpha_1, \ldots, \alpha_n \in Fm$,

$$\Gamma \cup \{\alpha_1, \ldots, \alpha_n\} \text{ is inconsistent in } \mathcal{I}\ell \iff \Gamma \vdash_{\mathcal{I}\ell} \neg(\alpha_1 \wedge \cdots \wedge \alpha_n).$$

Definition (extending RAFTERY's terminology)

A sequence $\langle \Psi_n(x_1, \ldots, x_n) : n \geqslant 1 \rangle$ of finite sets defines an **Inconsistency Lemma** for a generalized matrix $\langle \boldsymbol{A}, \mathscr{C} \rangle$ when for all $X \cup \{a_1, \ldots, a_n\} \subseteq A$,

$$X \cup \{a_1, \ldots, a_n\} \text{ is } C\text{-inconsistent} \iff \Psi_n^{\boldsymbol{A}}(a_1, \ldots, a_n) \subseteq C(X).$$

(C is the closure operator associated with the closure system $\mathscr{C}$.)

$\langle \{\neg(x_1 \wedge \cdots \wedge x_n)\} : n \geqslant 1 \rangle$ defines an Inconsistency Lemma for $\mathcal{I}\ell$.

A Tarski-style condition: the Inconsistency Lemma

Fact (essentially, Font and Jansana, 1996)

The Inconsistency Lemma does not transfer, in general, from a logic to arbitrary algebras; not even for finitary logics. Counterexample: $\mathcal{I}\ell_{\neg,\wedge}$

Theorem (Raftery, 2013)

Let $\mathscr{L}$ be a finitary and protoalgebraic logic.

The following conditions are equivalent.

(i) $\mathscr{L}$ satisfies an Inconsistency Lemma.

(ii) For every algebra $\boldsymbol{A}$, the generalized matrix $\langle \boldsymbol{A}, \mathcal{F}i_{\mathscr{L}}\boldsymbol{A} \rangle$ satisfies the same Inconsistency Lemma.

(iii) For all $\boldsymbol{A}$, the join-semilattice $\mathcal{F}i^{\omega}_{\mathscr{L}}\boldsymbol{A}$ is dually pseudo-complemented.

(iv) The join-semilattice $\mathcal{T}h^{\omega}\mathscr{L}$ is dually pseudo-complemented.

A Tarski-style condition: the Inconsistency Lemma

Theorem (Raftery, 2013)

Let $\mathscr{L}$ be a finitary and finitely algebraizable logic, and let the quasivariety K be its equivalent algebraic semantics.

The following conditions are equivalent.

(i) $\mathscr{L}$ satisfies an Inconsistency Lemma.

(iii) For all $\boldsymbol{A}$, the join-semilattice $\mathrm{Con}^{\omega}_{\mathsf{K}}\boldsymbol{A}$ is dually pseudo-complemented.

(v) For all $\boldsymbol{A} \in \mathsf{K}$, the join-semilattice $\mathrm{Con}^{\omega}_{\mathsf{K}}\boldsymbol{A}$ is dually pseudo-complemented.

Two general transfer results

Theorems (Czelakowski and Pigozzi, 2001,2004)

Let $\mathscr{L}$ be a finitary and protoalgebraic logic. Then:

1. Every property of a logic expressible by a first-order formula $\boldsymbol{\alpha}$ of the language of lattices transfers from $\mathscr{L}$ to all algebras; i.e.,

   $$\langle \mathcal{T}h\mathscr{L}, \cap, \vee \rangle \vDash \boldsymbol{\alpha} \implies \langle \mathcal{F}i_{\mathscr{L}}\boldsymbol{A}, \cap, \vee \rangle \vDash \boldsymbol{\alpha} \ \text{ for all } \boldsymbol{A}$$

2. Every property of a logic expressible by an accumulative set $\boldsymbol{G}$ of Gentzen-style rules transfers from $\mathscr{L}$ to all algebras; i.e.,

   $$\langle \boldsymbol{Fm}, \mathcal{T}h\mathscr{L} \rangle \text{ satisfies } \boldsymbol{G} \implies \langle \boldsymbol{A}, \mathcal{F}i_{\mathscr{L}}\boldsymbol{A} \rangle \text{ satisfies } \boldsymbol{G} \ \text{ for all } \boldsymbol{A}$$

A set $\boldsymbol{G}$ of Gentzen-style rules is **accumulative** when

$$\frac{\{\Gamma_i \rhd \varphi_i : i \in I\}}{\Gamma \rhd \varphi} \in \boldsymbol{G} \quad \implies \quad \frac{\{\Delta, \Gamma_i \rhd \varphi_i : i \in I\}}{\Delta, \Gamma \rhd \varphi} \in \boldsymbol{G}$$

The transfer problem of the strong property of congruence

Definition

Let $\langle \boldsymbol{A}, \mathscr{C} \rangle$ be a generalized matrix. The **Frege relation** of $F \subseteq A$ is the relation on A defined as follows: For every $a, b \in A$,

$$a \equiv b \ (\boldsymbol{\Lambda}^{A}_{\mathscr{C}} F) \overset{\text{def}}{\iff} C\big(F \cup \{a\}\big) = C\big(F \cup \{b\}\big).$$

$\langle \boldsymbol{A}, \mathscr{C} \rangle$ has the **strong property of congruence** when for every $F \in \mathscr{C}$, the Frege relation $\boldsymbol{\Lambda}^{A}_{\mathscr{C}} F$ is a congruence of the algebra $\boldsymbol{A}$.

$\langle \boldsymbol{Fm}, \mathcal{T}h\mathscr{L} \rangle$ satisfies the strong property of congruence $\iff$ $\mathscr{L}$ satisfies the **strong property of replacement**: for all $\Gamma \in \mathcal{T}h\mathscr{L}$ and all $\alpha, \beta \in Fm$,

if $\Gamma, \alpha \dashv\vdash_{\mathscr{L}} \Gamma, \beta$ then $\Gamma, \delta(\alpha, \vec{z}) \dashv\vdash_{\mathscr{L}} \Gamma, \delta(\beta, \vec{z})$ for all $\delta(x, \vec{z}) \in Fm$.

$\langle \boldsymbol{Fm}, \mathcal{T}h\mathscr{L} \rangle$ has the property $\Longrightarrow$ $\langle \boldsymbol{A}, \mathcal{F}i_{\mathscr{L}} \boldsymbol{A} \rangle$ has the property, for all $\boldsymbol{A}$ **?**

The transfer problem of the strong property of congruence

Fact (Bou, 2002; Babyonischev, 2003)

The strong property of congruence does not transfer in general, not even for finitary and truth-equational logics.

Theorem (Czelakowski and Pigozzi, 2004)

The strong property of congruence transfers for finitary and protoalgebraic logics.

Does the strong property of congruence transfer for non-finitary protoalgebraic logics ?

The transfer problem of the strong property of congruence

Theorem (Albuquerque, Font, Jansana, Moraschini, 2016)

The strong property of congruence transfers for fully selfextensional logics with theorems.

Does the strong property of congruence transfer for theorem-less fully selfextensional logics ?

(**Fully selfextensional logics** form one of the classes in the **Frege hierarchy** (slide 8); actually, a particularly well-behaved class.)

The set of all protoalgebraic logics (over a fixed language)

Language with at least one connective of arity 2 or greater

(**Fin**)**Log** := {(finitary) logics over this language}

(**Fin**)**Prot** := {(finitary) protoalgebraic logics over this language}

Facts

1. (**Fin**)**Log** is a complete lattice, ordered by the extension relation:

$$\mathscr{L} \leqslant \mathscr{L}' \stackrel{\text{def}}{\iff} \vdash_{\mathscr{L}} \subseteq \vdash_{\mathscr{L}'}$$

   Hence (**Fin**)**Prot** is an ordered set, under this relation.

2. (**Fin**)**Prot** is an up-set of (**Fin**)**Log**:

$$\mathscr{L} \text{ protoalgebraic }, \mathscr{L} \leqslant \mathscr{L}' \implies \mathscr{L}' \text{ protoalgebraic}$$

   Hence (**Fin**)**Prot** is a join-complete sub-semilattice of (**Fin**)**Log**, and has a maximum (the inconsistent logic).

What about the lower order structure of (Fin)Prot ?

Main results on the order in (**Fin**)**Prot**

Theorems

1. (**Fin**)**Prot** has no minimum.
2. (**Fin**)**Prot** is not a meet-semilattice.
3. (**Fin**)**Prot** has infinitely many strictly decreasing infinite sequences with no lower bound.
4. [Jansana] Every finite Boolean lattice is isomorphic to a lattice of logics in **FinProt**.
5. If $\mathscr{L} \in$ (**Fin**)**Prot** has a coherent set of protoimplication formulas, then $\mathscr{L}$ is not a minimal element of (**Fin**)**Prot**.

Protoimplication formulas and coherent sets

Theorem

A logic $\mathscr{L}$ is **protoalgebraic** if and only if it has a set $\Delta(x,y)$ of **protoimplication formulas**, i.e., such that:

$$\vdash_{\mathscr{L}} \Delta(x,x) \qquad (\mathrm{R}_\Delta)$$

$$x, \Delta(x,y) \vdash_{\mathscr{L}} y \qquad (\mathrm{MP}_\Delta)$$

Definition

A non-empty $\Delta(x,y)$ is **coherent** when for all $\delta, \delta' \in \Delta(x,y)$,

$$\delta(x,x) = \delta'(x,x).$$

- All the formulas in a coherent set have the same complexity.
- Coherent sets are finite.
- There are coherent sets of all finite cardinalities and all complexities.

The family of logics $\mathcal{I}\Delta$ for coherent $\Delta(x,y)$

Definition

Let $\Delta(x,y)$ be a coherent set.

The logic $\mathcal{I}\Delta$ is the logic defined by the axiomatic system with:

$$\text{the axiom} \quad \delta(x,x) \quad \text{for any } \delta(x,y) \in \Delta(x,y) \qquad (\mathrm{R}_\Delta)$$

$$\text{and the rule} \quad x, \Delta(x,y) \vdash y. \qquad (\mathrm{MP}_\Delta)$$

Theorem

The theorems of $\mathcal{I}\Delta$ are the formulas $\delta(\alpha,\alpha)$ for $\delta(x,y) \in \Delta(x,y)$ and any $\alpha \in Fm$.

Their complexity is $\geqslant$ the complexity of $\delta(x,x)$.

- There is no minimum $\mathscr{L} \in$ (**Fin**)**Prot**.
- There are many pairs $\mathscr{L}, \mathscr{L}' \in$ (**Fin**)**Prot** with no common theorems.

The family of logics $\mathcal{I}\Delta$, for coherent $\Delta(x,y)$

The **iteration** operation:

- $\delta(x,y) \longmapsto \delta^{\mathbf{i}}(x,y) := \delta\big(\delta(x,x),\delta(x,y)\big)$
- $\Delta(x,y) \longmapsto \Delta^{\mathbf{i}}(x,y) := \big\{\delta'\big(\delta(x,x),\delta(x,y)\big) : \delta,\delta' \in \Delta\big\}$

Theorems

1. $\Delta(x,y)$ coherent $\implies$ $\Delta^{\mathbf{i}}(x,y)$ coherent
2. $\mathcal{I}\Delta^{\mathbf{i}} < \mathcal{I}\Delta$

- If $\mathscr{L} \in$ (**Fin**)**Prot** has a coherent set of protoimplication formulas, then $\mathscr{L}$ is not a minimal element of (**Fin**)**Prot**.
- (**Fin**)**Prot** has infinitely many strictly decreasing infinite sequences with no lower bound.

Thank you !