

Interpreting Sequent Calculi as Client–Server Games

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Background

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- to breathe life into the resource metaphor, we need dynamics
⇒ game semantics for substructural sequent calculi

Different types of game semantics

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- (1) “propositions as games / connectives as game operators”
(since 1990s: Blass, Abramsky, Jagadeesan, Hyland, Ong, . . .)
 - **abstract** semantic models of (fragments and variants) of linear logic
 - leads to a **fully abstract semantic model** of PCF
- (2) “logical dialogue games”
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 - **Proponent/Opponent** games with **logical** and **structural** rules
 - proofs are **winning strategies** for Proponent

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- (3) Client/Server games (**C/S**-games)

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- a **client C** seeks to **extract/reconstruct** an IP H with respect to a whole **bunch of IPs** G_1, \dots, G_n maintained by the **server S**:
Notation: $G_1, \dots, G_n \triangleright H$

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- extraction proceeds **stepwise**, in **rounds**, initiated by **C**
- **C succeeds (wins)** if H is atomic and $\in \{G_1, \dots, G_n\}$ the **final state**. We are interested in **winning strategies** for **C**.

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in each state $\Gamma \triangleright H$ the client **C** may request one of two actions from **S**:

- **UNPACK** one of your (**S**'s) IP
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UNPACK-rules: **C** picks $G \in \Gamma$ (= bunch of IPs provided by **S**)

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(U_{\perp}^+) $G = \perp$: game ends, **C** wins

CHECK-rules: depend on **C**'s current IP H .

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(C_{given}) $H = (F_1 \text{ given } F_2)$: **S** adds F_2 to Γ , F_1 replaces H

(C_{atom}^+) H is atomic: game ends, **C** wins if $H \in \Gamma$

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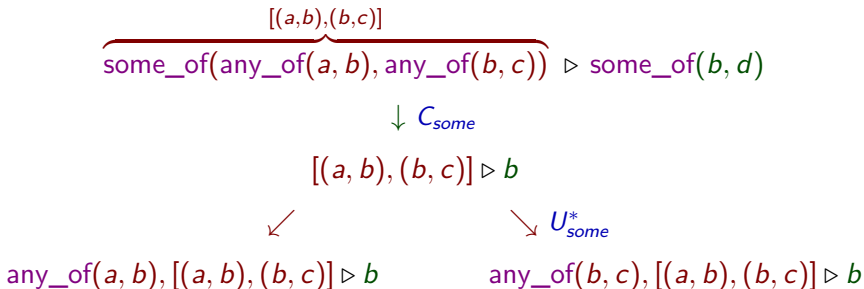
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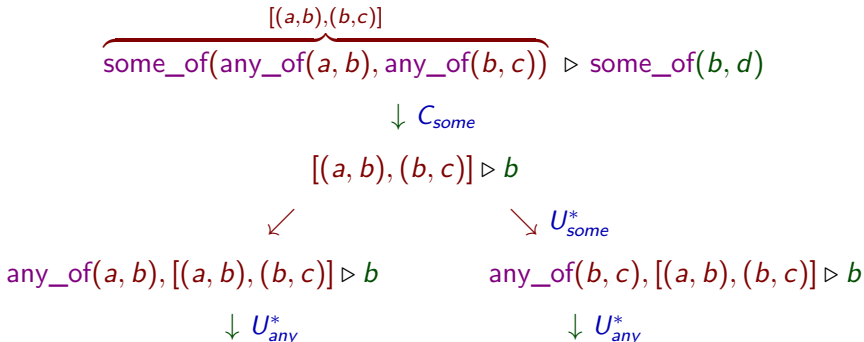
\swarrow

$\searrow U_{\text{some}}^*$

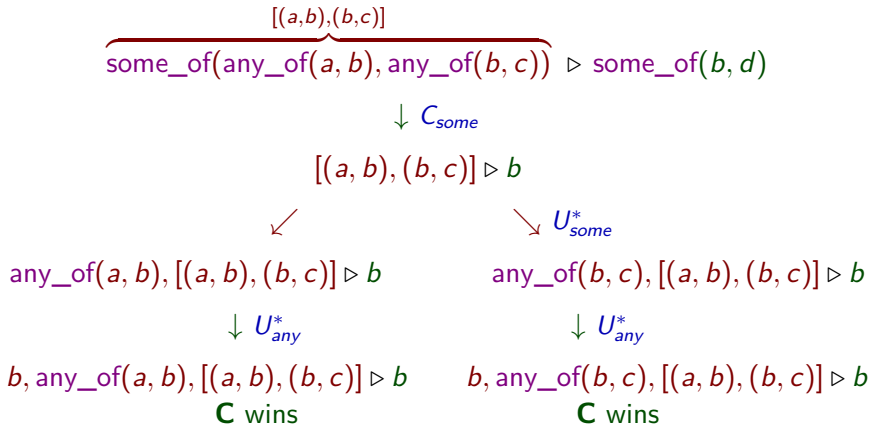
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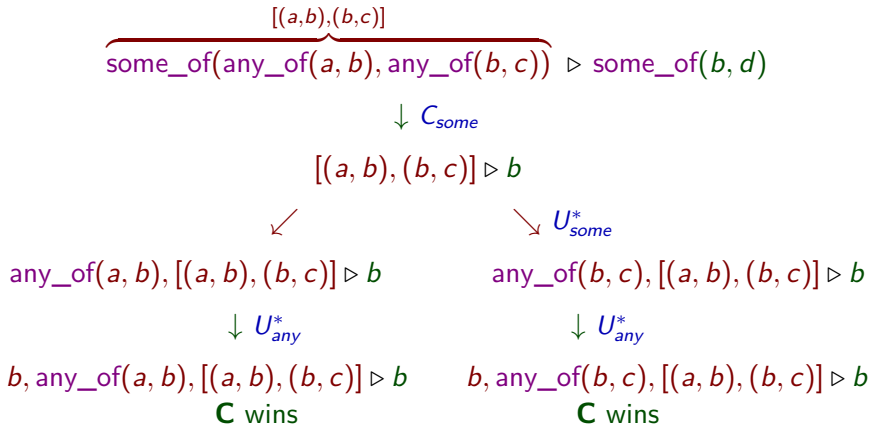
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Note: (winning) strategies for **C** are trees of states that branch for all choices of **S**

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- **some_of**(F_1, \dots, F_n) corresponds to $F_1 \vee \dots \vee F_n$
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Sequent calculus proofs in disguise

C's winning strategy for $[(a, b), (b, c)] \triangleright \text{some_of}(b, d)$ corresponds to

$$\frac{\frac{b, a \wedge b, (a \wedge b) \vee (b \wedge c) \vdash b}{a \wedge b, (a \wedge b) \vee (b \wedge c) \vdash b} (\wedge, l) \quad \frac{b, a \wedge b, (a \wedge b) \vee (b \wedge c) \vdash b}{a \wedge b, (a \wedge b) \vee (b \wedge c) \vdash b} (\wedge, l)}{(a \wedge b) \vee (b \wedge c) \vdash b} (\vee, l)}{(a \wedge b) \vee (b \wedge c) \vdash b \vee d} (\vee, r)$$

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Note:

- intuitionistic rules
- no structural rules

Gentzen's original LI/LK

Initial sequents: $A \vdash A$

Cut rule: $\frac{\Gamma \vdash \Delta, A \quad A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}$ (*cut*)

Structural rules:

$$\frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A} (w, r) \quad \frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} (w, l) \quad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A} (c, r) \quad \frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta} (c, l)$$

Logical rules:

$$\frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A} (\neg, r) \quad \frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta} (\neg, l)$$
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Llp – a proof search friendly version of LI:

- Initial sequents: $A, \Gamma \vdash \Delta, A / \perp, \Gamma \vdash \Delta \Rightarrow$ no weakening
- contraction built into logical rules, cut-free

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Corollary to the (cut-free!) soundness and completeness of **Llp**:

Theorem

C has a winning strategy for $G_1, \dots, G_n \triangleright F$ iff
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intuitionistic logic is hardly 'substructural'
 \Rightarrow find versions of the game that model resource consciousness

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- \Rightarrow contraction free intuitionistic logic

Weaking as explicit dismissal

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(*Dismiss*) **C** chooses $F \in \Gamma$, **S** removes F from Γ
- corresponds to **weakening** (w, l) of **LI**

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 - ▶ **replace** **arbitrary_many**(F) by F
 - ▶ add another **copy** of **arbitrary_many**(F)

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- **arbitrary_many**(F) corresponds to $!F$ of **linear logic**
- **dismissing**, **copying**, and **replacing** correspond to

$$\frac{\Gamma \vdash \Delta}{!A, \Gamma \vdash \Delta} (w!) \quad \frac{!A, !A, \Gamma \vdash \Delta}{!A, \Gamma \vdash \Delta} (c!) \quad \frac{A, \Gamma \vdash \Delta}{!A, \Gamma \vdash \Delta} l!$$

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S chooses whether to continue with $\Gamma_1 \triangleright F_1$ or $\Gamma_2 \triangleright F_2$
- to obtain a **C/S**-game for **full intuitionistic linear logic (ILL)**:
 - replace (U_{given}) by a 'splitting version' of it
 - C** can always add \emptyset (**empty IP** – corresponding to Girard's **1**) to **S**'s Γ
 - modify the **winning conditions**:
C wins in the following states: $A \triangleright A \quad \perp, \Gamma \triangleright A \quad \triangleright \emptyset$

Interpreting Lambek's calculus: sequences of IPs instead of multisets

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- the 'bunch of information' provided by **S** might be a list (sequence)
- if **S CHECKS** an conditional IP of **C**, the 'conditioning IP' is added either **first** or **last**:
 $\Rightarrow F_1$ **given** F_2 splits into F_1 **given** $\searrow F_2$, F_1 **given** $\nearrow F_2$ corresponding to

$$\frac{A, \Gamma \vdash B}{\Gamma \vdash A \searrow B} (\searrow, r) \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash B / A} (/ , r)$$

- UNPACKING** conditional information provided by **S** follows

$$\frac{\Gamma \vdash A \quad \Pi, B, \Sigma \vdash \Delta}{\Pi, \Gamma, A \searrow B, \Sigma \vdash \Delta} (\searrow, l) \qquad \frac{\Gamma \vdash A \quad \Pi, B, \Sigma \vdash \Delta}{\Pi, A / B, \Gamma, \Sigma \vdash \Delta} (/ , l)$$

- combined with a 'sequence version of conjunction' (fusion) this leads to an **C/S**-game for full Lambek calculus **FL**

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Topics for further investigation

- interpreting multi-conclusion calculi, in particular full **LL**
- systematic connections to other game semantics
- hypersequent systems modeled by parallel games
- ...