On Minimal Models for Horn Clauses over Predicate Fuzzy Logics

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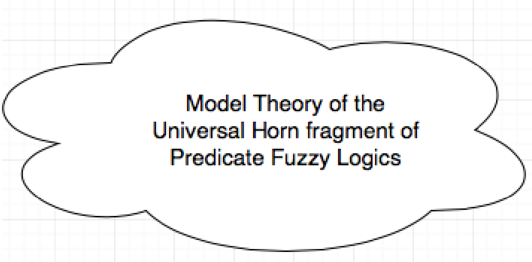
Main Objective



- Study of Horn clauses.
- Minimal models for universal Horn theories.
- Characterization of these minimal models by using Herbrand structures.

Importance

Logic programs allow a procedural interpretation, because there is a unique "generic" mathematical structure in which to interprete logic programs.

J.A. Makowsky.

Why Horn Formulas Matter in Computer Science: Initial Structures and Generic Examples. Journal of Computer and System Science, 34:266–292, 1987.

Horn clauses

- Introduction: McKinsey (1943).
- Good logic properties.
- Logic programming, abstract specification of data structures and relational data bases, abstract algebra and model theory.

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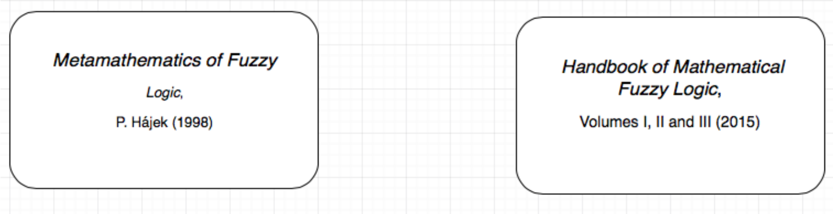

Horn clauses

Basic Horn Formula:

$\alpha_1 \& \cdots \& \alpha_n \to \beta$, where α_i, β are atomic formulas for $1 \leq i \leq m$.

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Horn clause

$(\forall x_0) \cdots (\forall x_n) \psi$,where ψ is a quantifier-free Horn formula.

Horn clauses

- In general, there are not equivalent disjunctive forms.

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- Weak Horn clauses and Strong Horn clauses.

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(Graded syntax)

State of the Art

Propositional Logic

- Borgwardt, Cerami and Peñaloza (2014)

$$\langle p_1 \& \ldots \& p_k \to q_1 \& \ldots \& q_m \geq r \rangle$$

$$\langle p_1 \& \ldots \& p_k \to \overline{0} \geq r \rangle$$

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First-order logic:

- Vychodil and Belohlávek (2005)

$$\bigwedge_{i=1}^{n=1} (t_i \approx t_i') \to t \approx t'$$

Preliminaries: Definitions

Definition

We define an $\mathbf{A}$-*structure* $\mathbf{M}$ for $\mathcal{P}$ as the triple $\langle M, (P_M)_{P \in Pred}, (F_M)_{F \in Func} \rangle$, where M is a nonempty domain, $P_{\mathbf{M}}$ is an n-ary fuzzy relation and $F_{\mathbf{M}}$ is a function from M^n to M.

Preliminaries: Definitions

Definition

If $\mathbf{M}$ is an $\mathbf{A}$-structure and v is an $\mathbf{M}$-evaluation, we define the *values* of terms and the *truth values* of formulas in M for an evaluation v recursively as follows:

$$||x||^{\mathbf{A}}_{\mathbf{M},v} = v(x);$$
$$||F(t_1, \ldots, t_n)||^{\mathbf{A}}_{\mathbf{M},v} = F_{\mathbf{M}}(||t_1||^{\mathbf{A}}_{\mathbf{M},v}, \ldots, ||t_n||^{\mathbf{A}}_{\mathbf{M},v});$$
$$||P(t_1, \ldots, t_n)||^{\mathbf{A}}_{\mathbf{M},v} = P_{\mathbf{M}}(||t_1||^{\mathbf{A}}_{\mathbf{M},v}, \ldots, ||t_n||^{\mathbf{A}}_{\mathbf{M},v});$$
$$||(\forall x)\varphi||^{\mathbf{A}}_{\mathbf{M},v} = inf\{||\varphi||^{\mathbf{A}}_{\mathbf{M},v[x\to a]} \mid a \in M\};$$
$$||(\exists x)\varphi||^{\mathbf{A}}_{\mathbf{M},v} = sup\{||\varphi||^{\mathbf{A}}_{\mathbf{M},v[x\to a]} \mid a \in M\}.$$

Preliminaries: Definitions

Definition

(f, g) *homomorphism* from $\langle \mathbf{A}, \mathbf{M} \rangle$ to $\langle \mathbf{B}, \mathbf{N} \rangle$ if
f is a homomorphism of L-algebras and

$$g(F_{\mathbf{M}}(d_1, \ldots, d_n)) = F_{\mathbf{N}}(g(d_1), \ldots, g(d_n))$$

If $P_{\mathbf{M}}(d_1, \ldots, d_n) = 1$, then $P_{\mathbf{N}}(g(d_1), \ldots, g(d_n)) = 1$.

Preliminaries: Definitions

Fuzzy equality $\approx$:

Preliminaries: Definitions

Fuzzy equality $\approx$:

- Equivalence relation.
- Axiom C1:

$$(\forall x_1)\cdots(\forall x_n)(\forall y_1)\cdots(\forall y_n)(x_1 \approx y_1 \& \cdots \& x_n \approx y_n \to F(x_1,\ldots,x_n) \approx F(y_1,\ldots,y_n))$$

- Axiom C2:

$$(\forall x_1)\cdots(\forall x_n)(\forall y_1)\cdots(\forall y_n)(x_1 \approx y_1 \& \cdots \& x_n \approx y_n \to (P(x_1,\ldots,x_n) \leftrightarrow P(y_1,\ldots,y_n)))$$

Minimal models for universal Horn theories.

Definition

Let Φ be a consistent theory, we define a binary relation on the set of terms, denoted by $\sim$, in the following way: for every terms t_1, t_2,

$$t_1 \sim t_2 \text{ if and only if } \Phi \vdash t_1 \approx t_2.$$

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$\sim$ is an equivalence relation compatible with the symbols of the language.

Minimal models for universal Horn theories.

Definition (Term Structure)

Let Φ be a consistent theory. We define the following structure $\langle \mathbf{B}, \mathbf{T}^{\Phi} \rangle$, where $\mathbf{B}$ is the two-valued Boolean algebra, $\mathbf{T}^{\Phi}$ is the set of all equivalence classes of the relation $\sim$ and

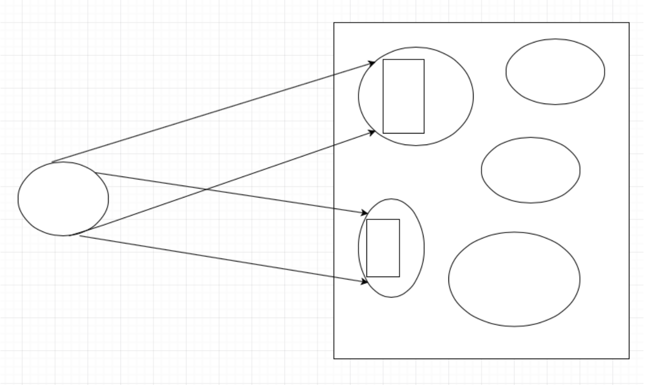
$$F_{\mathbf{T}^{\Phi}}(\overline{t_1}, \ldots, \overline{t_n}) = \overline{F(t_1, \ldots, t_n)}$$

$$||P(\overline{t_1}, \ldots, \overline{t_n})||^{\mathbf{B}}_{\mathbf{T}^{\Phi}} = \begin{cases} 1, & \text{if } \Phi \vdash P(t_1, \ldots, t_n) \\ 0, & \text{otherwise} \end{cases}$$

We call $\langle \mathbf{B}, \mathbf{T}^{\Phi} \rangle$ the *term structure associated to* Φ.

Minimality for models: Free Models

Free: unique homomorphism extending the assignation for variables.



Minimality for models: *A*-generic Models

Definition

Let $\mathbf{K}$ be a class of structures. Given $\langle \mathbf{B}, \mathbf{N} \rangle \in \mathbf{K}$, we say that $\langle \mathbf{B}, \mathbf{N} \rangle$ is A-*generic in* $\mathbf{K}$ if for every atomic sentence φ:

$||\varphi||^{\mathbf{B}}_{\mathbf{N}} = 1$ if and only if for every structure $\langle \mathbf{A}, \mathbf{M} \rangle \in \mathbf{K}$, $||\varphi||^{\mathbf{A}}_{\mathbf{M}} = 1$.

Minimality for models: *A*-generic Models

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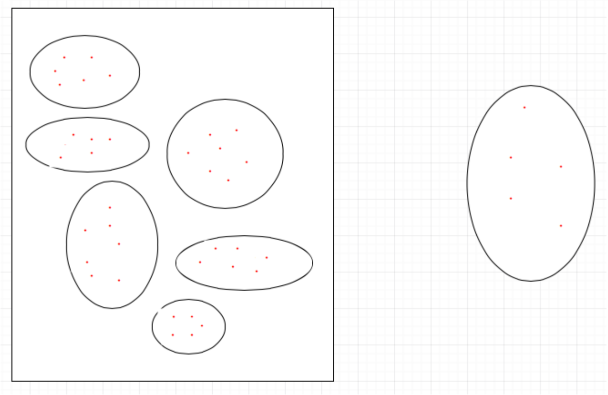

Minimal models for universal Horn theories.

Definition

Let e^{Φ} be the following $\mathbf{T}^{\Phi}$-evaluation: $e^{\Phi}(x) = \overline{x}$.

The term structure is A-generic:

Lemma

Let Φ be a consistent theory, and φ any atomic formula,

$$||\varphi||^{\mathbf{B}}_{\mathbf{T}^{\Phi}, e^{\Phi}} = 1 \textit{ if and only if } \Phi \vdash \varphi.$$

Minimal models for universal Horn theories.

The term structure is free:

Theorem

Let Φ be a consistent theory with $||\Phi||^{\mathbf{B}}_{\mathbf{T}^{\Phi},e^{\Phi}} = 1$. Then, for every reduced structure $\langle \mathbf{A}, \mathbf{M} \rangle$ and every evaluation v such that $||\Phi||^{\mathbf{A}}_{\mathbf{M},v} = 1$, there is a unique homomorphism (f, g) from $\langle \mathbf{B}, \mathbf{T}^{\Phi} \rangle$ to $\langle \mathbf{A}, \mathbf{M} \rangle$ such that for every $x \in Var$, $g(\overline{x}) = v(x)$.

Minimal models for universal Horn theories.

Sketch of the proof:

- Homomorphism: $(id_{\mathbf{B}}, g)$, where $g : T^{\Phi} \to M$ is defined as:
  $$g(\bar{t}) = ||t||^{\mathbf{A}}_{\mathbf{M},v} \text{ for every term } t.$$

Minimal models for universal Horn theories.

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Minimal models for universal Horn theories.

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- Homomorphism: $(id_{\mathbf{B}}, g)$, where $g : T^{\Phi} \to M$ is defined as:
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- g is well-defined because any $\langle \mathbf{A}, \mathbf{M} \rangle$ is <u>reduced</u>.
- Unicity: $\{\bar{x} \mid x \in Var\}$ generates the universe T^{Φ}.

Minimal models for universal Horn theories.

Remark

If the similarity is interpreted as the crisp equality, $\langle \mathbf{B}, \mathbf{T}^{\Phi} \rangle$ is free on the class of all models of the associated theory Φ.

Minimal models for universal Horn theories.

Not every term structure associated to a consistent theory is a model of the theory:

If $\Phi = \{\neg(\overline{1} \to P(a)) \& \neg(P(a) \to \overline{0})\}$, then

$$||\Phi||^{\mathbf{B}}_{\mathbf{T}^{\Phi}, e^{\Phi}} \neq 1$$

Minimal models for universal Horn theories.

Definition

We define the rank of a formula φ $rk(\varphi)$ recursively as:

- $rk(\varphi) = 0$, if φ is atomic;
- $rk(\neg\varphi) = rk((\exists x)\varphi) = rk((\forall x)\varphi) = rk(\varphi) + 1$;
- $rk(\varphi \circ \psi) = rk(\varphi) + rk(\psi)$, for every binary propositional connective $\circ$.

Minimal models for universal Horn theories.

In general, in fuzzy logics:

$$\forall x(\varphi \& \psi) \not\equiv (\forall x)\varphi \& (\forall x)\psi$$

Then, strong Horn clauses are not recursively definable.

Therefore, we use induction on the rank of Horn clauses (not on the complexity of the clauses).

Minimal models for universal Horn theories.

Theorem

Let Φ be a consistent theory. For every Horn clause φ, if $\Phi \vdash \varphi$, then $||\varphi||^{\mathbf{B}}_{\mathbf{T}^{\Phi}, e^{\Phi}} = 1$.

Minimal models for universal Horn theories.

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Sketch of the proof:

By induction on the rank of the Horn clause φ.

Minimal models for universal Horn theories.

G$\forall$, $\Phi = \{\neg(P\overline{c} \rightarrow \overline{0})\}$, $\varphi = P\overline{c} \rightarrow \overline{0}$ and using the A-genericity.

(Details here)

Herbrand structures

- The theory Φ is $\approx$-free.
- Some works: Cintula and Metcalfe (2013) and Gerla (2005, fuzzy logic programming).
- H-structure: a particular case of Herbrand structure. We define intersections of H-structures.
- Among other results, we proved a characterization of minimal models of equality-free Horn clauses without free variables:

Theorem

Let $\mathbf{K}$ *be the class of all models of a consistent set of equality-free sentences which are Horn clauses. The intersection of the family of all H-structures in* $\mathbf{K}$ *is the free model in* $\mathbf{K}$.

Sketch of the proof here.

Minimality.

Definition

A structure $\langle \mathbf{B}, \mathbf{N} \rangle$ is a *fully named model* if for any element n of the domain N, there exists a ground term t such that $||t||^{\mathbf{B}}_{\mathbf{N}} = n$.

(Example: Herbrand structures)

Theorem

Let $\mathbf{K}$ *be a class of structures and* $\langle \mathbf{B}, \mathbf{M} \rangle \in \mathbf{K}$ *be a fully named model with* $\mathbf{B} = \mathbf{F}_{\mathbf{MTL}}(\overline{\emptyset})$. *Then,*

$$\langle \mathbf{B}, \mathbf{M} \rangle \text{ is free in } \mathbf{K} \text{ if and only if } \langle \mathbf{B}, \mathbf{M} \rangle \text{ is A-generic in } \mathbf{K}.$$

Sketch of the proof here.

Future Work

Fuzzy Basic Horn Formula:

$(\alpha_1, r_1)\& \cdots \&(\alpha_n, r_n) \to (\beta, s)$, where $(\alpha_1, r_1) \ldots, (\alpha_n, r_n), (\beta, s)$

- Term structure associated to a consistent set of senteces $\langle \mathbf{B}, \mathbf{T}^{\Phi} \rangle$.
- $\langle \mathbf{B}, \mathbf{T}^{\Phi} \rangle$ is A-generic and free on the class of reduced models of Φ.

Open problem: generalization of the results concerning to fuzzy Horn clauses to fuzzy logics with enriched language whenever it is possible.

Thank you!

Appendix

Definition

A binary left-continuous function $* : [0,1]^2 \to [0,1]$ is a *left-continuous t-norm* if it is commutative, associative, monotone and 1 is its unit element.

Definition

Given a left-continuous t-norm $*$, its residuum is defined as $x \Rightarrow y = sup\{z \in [0,1] \mid x * z \leq y\}$ for $x, y \in [0,1]$.

Back.

Appendix

Lemma

Let Φ be a theory. If for every $1 \leq i \leq n$, $t_i \sim t'_i$, then

(i) $F(t_1, \ldots, t_n) \sim F(t'_1, \ldots, t'_n)$, *and*

(ii) $\Phi \vdash P(t_1, \ldots, t_n)$ *iff* $\Phi \vdash P(t'_1, \ldots, t'_n)$

Back.

Appendix

$$\text{G}\forall.\ \ \Phi = \{\neg(P\overline{c} \to \overline{0})\} \text{ and } \varphi = P\overline{c} \to \overline{0}.$$

$\Phi \not\models \varphi$: G-algebra $\mathbf{A}$, and $\langle \mathbf{A}, \mathbf{M} \rangle$ such that $||P\overline{c}||^{\mathbf{A}}_{\mathbf{M}} = 0.8$, then $||\Phi||^{\mathbf{A}}_{\mathbf{M}} = 1$ and $||P\overline{c} \to \overline{0}||^{\mathbf{A}}_{\mathbf{M}} \neq 1$ consequently $\Phi \not\models_G P\overline{c} \to \overline{0}$. With the same $\langle \mathbf{A}, \mathbf{M} \rangle$, $\Phi \not\models_G P\overline{c}$.

$||\varphi||^{\mathbf{B}}_{\mathbf{T}^\Phi} = 1$: Since $\Phi \not\models_G P\overline{c}$ is A-generic, $||P\overline{c}||^{\mathbf{B}}_{\mathbf{T}^\Phi} = 0$ and then $||\varphi||^{\mathbf{B}}_{\mathbf{T}^\Phi} = 1$.

Back.

Appendix

Definition

The *Herbrand universe of a predicate language* is the set of all ground terms of the language. A *Herbrand structure* is a structure $\langle \mathbf{A}, \mathbf{H} \rangle$, where $\mathbf{H}$ is the Herbrand universe, and:

For any individual constant symbol c, $c_{\mathbf{H}} = c$.

For any n-ary function symbol F and any $t_1, \ldots, t_n \in H$,

$$F_{\mathbf{H}}(t_1, \ldots, t_n) = F(t_1, \ldots, t_n)$$

Back.

Appendix

H-structure:

- **B**: the two-valued Boolean algebra
- For every $n \geq 1$ and every n-ary predicate symbol P,

$$P_{\mathcal{H}}(t_1, \ldots, t_n) = \begin{cases} 1, & \text{if } P(t_1, \ldots, t_n) \in H \\ 0, & \text{otherwise.} \end{cases}$$

Back.

Appendix

Definition

Let I be a nonempty set and for every $i \in I$, $H_i \subset \overline{H}$. We call $\langle \mathbf{B}, \mathbf{N}^H \rangle$ the *intersection* of the family of H-structures $\{\langle \mathbf{B}, \mathbf{N}^{H_i} \rangle \mid i \in I\}$, where $H = \bigcap_{i \in I} H_i$.

Back.

Appendix

Lemma

Assume that φ is an equality-free consistent sentence which is a Horn clause. If $\{\langle \mathbf{B}, \mathbf{N}^{H_i} \rangle \mid i \in I\}$ is the family of all H-models of φ and $H = \bigcap_{i \in I} H_i$, then $\langle \mathbf{B}, \mathbf{N}^{H} \rangle$ is also an H-model of φ.

Sketch of the proof here.

Corollary

An equality-free consistent sentence which is a Horn clause has a model if and only if it has an H-model.

Back.

Appendix

Sketch of the proof:

- Let $\langle \mathbf{A}, \mathbf{M} \rangle$ be a structure and H be the set of all atomic equality-free sentences σ such that $||\sigma||^{\mathbf{A}}_{\mathbf{M}} = 1$. Then, for every equality-free sentence φ which is an Horn clause, if $||\varphi||^{\mathbf{A}}_{\mathbf{M}} = 1$, then $||\varphi||^{\mathbf{B}}_{\mathbf{N}^H} = 1$, where $\langle \mathbf{B}, \mathbf{N}^H \rangle$ is an H-structure.
- Induction on the rank of φ.
- Let φ be a Horn clause where $x_1, \ldots, x_m$ are pairwise distinct free variables. Then, for every terms $t_1, \ldots, t_m$,

$$\varphi(t_1, \ldots, t_m / x_1, \ldots, x_m)$$

  is a Horn clause.

Appendix

Sketch of the proof:

- By The Model Intersection Property, the intersection of the family of all H-structures in $\mathbf{K}$ is also a member of $\mathbf{K}$.
- We shown that the intersection is an A-generic structure in $\mathbf{K}$.
- As we will see later, in this case $\Rightarrow$ A-genericity implies free on $\mathbf{K}$.

Back.

Appendix

Definition

A structure $\langle \mathbf{B}, \mathbf{N} \rangle$ is a *fully named model* if for any element n of the domain N, there exists a ground term t such that $||t||^{\mathbf{B}}_{\mathbf{N}} = n$.

Back.

Appendix

Sketch of the proof:

$\Rightarrow$:

- $\langle \mathbf{B}, \mathbf{M} \rangle$ is free in $\mathbf{K}$ and the homomorphism preserves atomic formulas ([Dellunde, García-Cerdaña and Noguera, 2016])

Appendix

$\Leftarrow$:

- The unique homomorphism between the algebras: Birkhoff's Theorem (universal mapping property).
- The homomorphism $g : N \rightarrow M$: $g(t_N) = t_M$ for any ground term.
- Unicity: by the definition of g.

Back.

Definition

Let Φ be a consistent theory of **sentences**, we define a binary relation on the set of terms, denoted by $\sim$, in the following way: for every terms t_1, t_2,

$$t_1 \sim t_2 \text{ if and only if } |t_1 \approx t_2|_\Phi = 1.$$

Back.

Appendix

Definition (Term structure)

Let Φ be a consistent theory of sentences and $\mathbf{B} = [0,1]_{\mathrm{RPL}}$. We define the following structure $\langle \mathbf{B}, \mathbf{T}^{\Phi} \rangle$, where T^{Φ} is the set of all equivalence classes of the relation $\sim$ and

- For any n-ary function symbol F,

$$F_{\mathbf{T}^{\Phi}}(\overline{t_1}, \ldots, \overline{t_n}) = \overline{F(t_1, \ldots, t_n)}$$

- For any n-ary predicate symbol P,

$$P_{\mathbf{T}^{\Phi}}(\overline{t_1}, \ldots, \overline{t_n}) = |P(t_1, \ldots, t_n)|_{\Phi}$$

We call $\langle \mathbf{B}, \mathbf{T}^{\Phi} \rangle$ the *term structure associated to* Φ.

Back.

Appendix

Lemma

Let Φ *be a theory of sentences, the following holds:*

(ii) *For any atomic formula* φ, $||\varphi||^{\mathbf{B}}_{\mathbf{T}^{\Phi}, e^{\Phi}} = 1$ *if and only if* $|\varphi|_{\mathbf{\Phi}} = 1$.

(iii) *For any evaluated atomic formula* φ, $||\varphi||^{\mathbf{B}}_{\mathbf{T}^{\Phi}, e^{\Phi}} = 1$ *if and only if* $|\varphi|_{\mathbf{\Phi}} = 1$.

Back.

Appendix

Theorem

Let Φ be a consistent theory of sentences such that $\langle [0,1]_{RPL}, \mathbf{T}^{\Phi} \rangle$ is a model of Φ. Then $\langle [0,1]_{RPL}, \mathbf{T}^{\Phi} \rangle$ is free on the class of the reduced $[0,1]_{RPL}$-models of Φ.

Back.