Neighborhood semantics for non-classical logics with modalities

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The goal of this presentation

To study neighborhood semantics for modal many-valued logics

Neighborhood semantics (Scott–Montague 1970) provides,
in the classical case, a possible-worlds semantics for logics where
the usual Kripke semantics is not adequate

In particular we will

- define it for a very wide class of logics
- describe its relation with Kripke-style semantics
- axiomatize global consequence relations (w.r.t. all models)

Future work:

- axiomatize global consequence relations w.r.t. classes of models (i.e. extensions with modal axioms)
- study local consequence relations

Modal many-valued logics – 1

Many-valued logics: logics complete w.r.t. an intended semantics of algebras with more than two truth-values (typically $\mathrm{FL_{ew}}$-algebras).

Modal many-valued logics: expansions of many-valued logics with modal operators

- Fitting 1992
- Hájek 1998
- Caicedo, Rodríguez 2010
- Bou, Esteva, Godo, Rodríguez 2011
- Marti, Metcalfe 2014
- Vidal 2015
- Caicedo, Metcalfe, Rodríguez, Rogger 2016
- Godo, Rodríguez 2016
- Cintula, Noguera, Rogger 2016

Modal many-valued logics – 2

Modal many-valued logics are endowed with a Kripke-style semantics based on a many-valued scale which provides:

1. truth-values of propositions at each possible world
2. degree of accessibility from one world to another.

Modal many-valued logics – 2

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1. truth-values of propositions at each possible world
2. degree of accessibility from one world to another.

Problems:

- Axiomatizing a Kripke-style semantics over a given algebra (or class of algebras) of truth-values is in general a difficult problem.
- Conversely, proof systems with natural syntactic conditions may fail to be complete with any such Kripke-style semantics.

Classical neighborhood semantics

SM-model: $\mathfrak{M} = \langle W, N, V \rangle$, where

- $W \neq \emptyset$ (worlds)
- $N \colon W \to 2^{2^W}$ (neighborhood function)
- $V \colon Var \times W \to 2$ (classical evaluation), extended to all formulas, in particular:

$$V^{\mathfrak{M}}(\Box\varphi, x) = 1 \qquad \text{iff} \qquad [\![\varphi]\!]_{\mathfrak{M}} \in N(x),$$

where $[\![\varphi]\!]_{\mathfrak{M}} = \{y \in W \mid V^{\mathfrak{M}}(\varphi, y) = 1\}$ the set of worlds where "φ is true".

$\varphi \in Fm_{\Box\Diamond}$ is valid in $\mathfrak{M}$ if $[\![\varphi]\!]_{\mathfrak{M}} = W$.

Global SM-consequence: $\Gamma \models_{\mathrm{SM}} \varphi$.

Classical Kripke semantics

K-model: $\mathcal{M} = \langle W, R, V\rangle$, where

- $W \neq \emptyset$ (worlds)
- $R \subseteq W^2$ (accessibility relation)
- $V\colon \mathit{Var} \times W \to 2$ (classical evaluation), extended to all formulas, in particular:

$$V^{\mathcal{M}}(\Box\varphi, x) = 1 \qquad \text{iff} \qquad R[x] \subseteq [\![\varphi]\!]_{\mathcal{M}}$$

where $R[x] = \{y \in W \mid Rxy\}$ and $[\![\varphi]\!]_{\mathcal{M}} = \{y \in W \mid V^{\mathcal{M}}(\varphi, y) = 1\}$.

K-validity and global K-consequence are defined analogously to SM-validity and SM-consequence.

Relation between classical neighborhood and Kripke

1. Given any K-model $\mathcal{M} = \langle W, R, V \rangle$, we obtain an SM-model $\mathfrak{M} = \langle W, N_R, V \rangle$ by setting for all $x \in W$,

$$X \in N_R(x) \qquad \text{iff} \qquad R[x] \subseteq X.$$

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$$X \in N_R(x) \qquad \text{iff} \qquad R[x] \subseteq X.$$

2. Given any SM-model $\mathfrak{M} = \langle W, N, V\rangle$, we can define a K-model $\mathcal{M} = \langle W, R_N, V\rangle$ by setting for all $x, y \in W$,

$$R_N xy \qquad \text{iff} \qquad y \in X, \text{ for each } X \in N(x).$$

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$$R_N xy \qquad \text{iff} \qquad y \in X, \text{ for each } X \in N(x).$$

To preserve the truth of all formulas at each world, $\mathfrak{M}$ has to be augmented, i.e., satisfy two additional conditions:

- $N(x)$ contains its core, i.e. the set $(\bigcap_{X \in N(x)} X) \in N(x)$,
- $N(x)$ is closed under taking supersets, i.e. if $X \in N(x)$ and $X \subseteq Y$, then $Y \in N(x)$.

Relation between classical neighborhood and Kripke

Theorem 1

(a) *Let* $\mathcal{M} = \langle W, R, V\rangle$ *be a* K*-model. Then* $\mathfrak{M} = \langle W, N_R, V\rangle$ *is an augmented* SM*-model,* $R_{N_R} = R$*, and* $V^{\mathfrak{M}} = V^{\mathcal{M}}$*.*

(b) *Let* $\mathfrak{M} = \langle W, N, V\rangle$ *be an augmented* SM*-model. Then* $\mathcal{M} = \langle W, R_N, V\rangle$ *is a* K*-model,* $N_{R_N} = N$*, and* $V^{\mathcal{M}} = V^{\mathfrak{M}}$*.*

$\models_{\mathrm{ASM}}$: semantical consequence of augmented SM-models.

Corollary 2

For any subset $\Gamma \subseteq Fm_{\Box\Diamond}$ *and formula* $\varphi \in Fm_{\Box\Diamond}$*,*

$$\Gamma \models_{\mathrm{ASM}} \varphi \quad \textit{iff} \quad \Gamma \models_{\mathrm{K}} \varphi.$$

The logic of classical neighborhood models

Let SM be the expansion of classical logic with

$$(E) \qquad \frac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$$

Theorem 3

Let $\Gamma \cup \{\varphi\} \subseteq Fm_{\Box\Diamond}$, *then*

$$\Gamma \vdash_{\mathrm{SM}} \varphi \qquad \textit{iff} \qquad \Gamma \models_{\mathrm{SM}} \varphi.$$

$\mathrm{FL}_{\mathrm{ew}}$-algebras

An $\mathrm{FL}_{\mathrm{ew}}$*-algebra* is a structure $\boldsymbol{A} = \langle A, \wedge, \vee, \&, \to, \overline{0}, \overline{1} \rangle$ such that:

(1) $\langle A, \wedge, \vee, \overline{0}, \overline{1} \rangle$ is a bounded lattice,

(2) $\langle B, \&, \overline{1} \rangle$ is a commutative monoid,

(3) $z \leq x \to y$ iff $x \& z \leq y$, (residuation)

Many-valued Kripke semantics (for a fixed complete $\mathrm{FL}_{\mathrm{e,w}}$-alg. $\boldsymbol{A}$)

$\mathrm{K}(\boldsymbol{A})$-model: $\mathcal{M} = \langle W, R, V \rangle$ such that

- $W \neq \emptyset$ (worlds)
- $R\colon W \times W \to A$ (accessibility relation)
- $V\colon Var \times W \to A$ (evaluation), extended to formulas inductively: using operations of $\boldsymbol{A}$ for the non-modal connectives and

$$V^{\mathcal{M}}(\Box\varphi, x) = \bigwedge\{Rxy \to V(\varphi, y) \mid y \in W\} = (R[x] \subseteq [\![\varphi]\!]_{\mathcal{M}}).$$

$$V^{\mathcal{M}}(\Diamond\varphi, x) = \bigvee\{Rxy \,\&\, V(\varphi, y) \mid y \in W\} = (R[x] \between [\![\varphi]\!]_{\mathcal{M}}).$$

where we define:

the A-valued subsets of W	to which y belongs to the degree
$R[x] = \{y \in W \mid Rxy\}$	it is accessible from x
$[\![\varphi]\!]_{\mathcal{M}} = \{y \in W \mid V^{\mathcal{M}}(\varphi, y)\}$	the formula φ is valid in x

A-valued neighborhood semantics (for a fixed $\mathrm{FL_{ew}}$-algebra $\boldsymbol{A}$)

$\mathrm{SM}(\boldsymbol{A})$-model: $\mathfrak{M} = \langle W, N^{\Box}, N^{\Diamond}, V\rangle$ such that

- $W \neq \emptyset$ (worlds)
- $N^{\Box}, N^{\Diamond} \colon W \to A^{A^W}$ (neighborhood functions)
- $V \colon Var \times W \to A$ (evaluation), extended to formulas inductively: using operations of $\boldsymbol{A}$ for the non-modal connectives and

$$V^{\mathfrak{M}}(\Box\varphi, x) = ([\![\varphi]\!]_{\mathfrak{M}} \in N^{\Box}(x)),$$

$$V^{\mathfrak{M}}(\Diamond\varphi, x) = ([\![\varphi]\!]_{\mathfrak{M}} \in N^{\Diamond}(x)).$$

where, as before, $[\![\varphi]\!]_{\mathfrak{M}} = \{y \in W \mid V^{\mathfrak{M}}(\varphi, y)\}$.

Augmented SM-frames

An $\mathrm{SM}(\boldsymbol{A})$-frame $\langle W, N^{\Box}, N^{\Diamond} \rangle$ is augmented if for each $x \in W$ there is an A-valued subset C_x of W (the core of $N^{\Box}(x)$ and $N^{\Diamond}(x)$) such that

$$(C_x \subseteq Y) = (Y \in N^{\Box}(x))$$

$$(C_x \between Y) = (Y \in N^{\Diamond}(x))$$

Relation between A-valued neighborhood and Kripke

1. Given a $\mathrm{K}(\boldsymbol{A})$-frame $\langle W, R\rangle$, we define an $\mathrm{SM}(\boldsymbol{A})$-frame $\langle W, N_R^{\Box}, N_R^{\Diamond}\rangle$:

$$
\begin{aligned}
N_R^{\Box}(x) &= \{X \in A^W \mid R[x] \subseteq X\}, \\
N_R^{\Diamond}(x) &= \{X \in A^W \mid R[x] \between X\}.
\end{aligned}
$$

Relation between A-valued neighborhood and Kripke

1. Given a $\mathrm{K}(\boldsymbol{A})$-frame $\langle W, R\rangle$, we define an $\mathrm{SM}(\boldsymbol{A})$-frame $\langle W, N_R^{\Box}, N_R^{\Diamond}\rangle$:

$$N_R^{\Box}(x) = \{X \in A^W \mid R[x] \subseteq X\},$$

$$N_R^{\Diamond}(x) = \{X \in A^W \mid R[x] \between X\}.$$

2. Given an $\mathrm{SM}(\boldsymbol{A})$-frame $\langle W, N^{\Box}, N^{\Diamond}\rangle$, we define two accessibility relations:

$$R_{N^{\Box}}(x, y) = (\forall X)(X \in N^{\Box}(x) \to y \in X),$$

$$R_{N^{\Diamond}}(x, y) = (\forall X)(y \in X \to X \in N^{\Diamond}(x)).$$

Relation between A-valued neighborhood and Kripke

1. Given a $\mathrm{K}(\boldsymbol{A})$-frame $\langle W, R\rangle$, we define an $\mathrm{SM}(\boldsymbol{A})$-frame $\langle W, N_R^{\Box}, N_R^{\Diamond}\rangle$:

$$N_R^{\Box}(x) = \{X \in A^W \mid R[x] \subseteq X\},$$
$$N_R^{\Diamond}(x) = \{X \in A^W \mid R[x] \between X\}.$$

2. Given an $\mathrm{SM}(\boldsymbol{A})$-frame $\langle W, N^{\Box}, N^{\Diamond}\rangle$, we define two accessibility relations:

$$R_{N^{\Box}}(x,y) = (\forall X)(X \in N^{\Box}(x) \to y \in X),$$
$$R_{N^{\Diamond}}(x,y) = (\forall X)(y \in X \to X \in N^{\Diamond}(x)).$$

Lemma 4

If $\langle W, N^{\Box}, N^{\Diamond}\rangle$ is augmented, then $\forall x \in W$, $C_x = R_{N^{\Box}}[x] = R_{N^{\Diamond}}[x]$.

Relation between A-valued neighborhood and Kripke

Theorem 5

(a) *Given a* $\mathrm{K}(\boldsymbol{A})$*-model* $\mathcal{M} = \langle W, R, V\rangle$*, define the augmented* $\mathrm{SM}(\boldsymbol{A})$*-model* $\mathfrak{M} = \langle W, N_R^{\Box}, N_R^{\Diamond}, V\rangle$*.*

Then $R_{N_R^{\Box}} = R_{N_R^{\Diamond}} = R$ *and for all* $\varphi \in Fm_{\Box\Diamond}$ *and all* $x \in W$*:*

$$V^{\mathfrak{M}}(\varphi, x) = V^{\mathcal{M}}(\varphi, x).$$

(b) *Given an augmented* $\mathrm{SM}(\boldsymbol{A})$*-model* $\mathfrak{M} = \langle W, N^{\Box}, N^{\Diamond}, V\rangle$*, define the* $\mathrm{K}(\boldsymbol{A})$*-model* $\mathcal{M} = \langle W, R = R_{N^{\Box}} = R_{N^{\Diamond}}, V\rangle$*.*

Then $N_R^{\Box} = N^{\Box}$*,* $N_R^{\Diamond} = N^{\Diamond}$*, and for all* $\varphi \in Fm_{\Box\Diamond}$ *and all* $x \in W$*:*

$$V^{\mathcal{M}}(\varphi, x) = V^{\mathfrak{M}}(\varphi, x).$$

Relation between A-valued neighborhood and Kripke

Corollary 6

For all subsets $\Gamma \cup \{\varphi\} \subseteq Fm_{\Box\Diamond}$,

$\Gamma \models_{\mathrm{K}(\boldsymbol{A})} \varphi$ *iff* $\mathfrak{M} \models_{\mathrm{SM}(\boldsymbol{A})} \varphi$ *for all augmented* $\mathrm{SM}(\boldsymbol{A})$*-models* $\mathfrak{M}$ *such that* $\mathfrak{M} \models_{\mathrm{SM}(\boldsymbol{A})} \Gamma$.

$\mathbb{K}$-valued neighborhood semantics ($\mathbb{K}$ a class of $\mathrm{FL_{ew}}$-algebras)

SM($\mathbb{K}$)-model: $\mathfrak{M} = \langle W, \langle \boldsymbol{A}_w \rangle_{w \in W}, N^{\Box}, N^{\Diamond}, V \rangle$ such that

- $W \neq \emptyset$ (worlds)
- $\boldsymbol{A}_w \in \mathbb{K}$ for each $w \in W$ (scales)
- $N^{\Box}, N^{\Diamond} \colon W \to (\bigcup_{v \in W} A_v)^{\prod_{v \in W} A_v}$ (neighborhood functions), such that $rg(N^{\Box}(w)), rg(N^{\Diamond}(w)) \subseteq A_w$
- $V \colon Var \times W \to \bigcup_{v \in W} A_v$ (evaluation), such that $V[Var \times \{w\}] \subseteq A_w$ and is extended to formulas inductively: using operations of $\boldsymbol{A}$ for the non-modal connectives and

$$V^{\mathfrak{M}}(\Box\varphi, x) \quad = \quad ([\![\varphi]\!]_{\mathfrak{M}} \in N^{\Box}(x)),$$

$$V^{\mathfrak{M}}(\Diamond\varphi, x) \quad = \quad ([\![\varphi]\!]_{\mathfrak{M}} \in N^{\Diamond}(x)).$$

where, as before, $[\![\varphi]\!]_{\mathfrak{M}} = \{x \in W \mid V^{\mathfrak{M}}(\varphi, x)\}$.

Completeness of a many-valued logic w.r.t. a class $\mathbb{K}$ of algebras

Let L be an axiomatic extension of $\mathrm{FL_{ew}}$ and $\mathbb{K}$ a class of L-algebras.

- L is strongly complete with respect to $\mathbb{K}$ if for every $\Gamma \cup \{\varphi\} \subseteq Fm$ we have: $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\Gamma \models_{\mathbb{K}} \varphi$.
- L is finitely strongly complete with respect to $\mathbb{K}$ if the same property holds for each *finite* set $\Gamma \cup \{\varphi\} \subseteq Fm$.
- L is complete with respect to $\mathbb{K}$ if for every $\Gamma \cup \{\varphi\} \subseteq Fm$ we have: $\vdash_{\mathrm{L}} \varphi$ iff $\models_{\mathbb{K}} \varphi$.

An axiomatization of the global logic of $\mathrm{SM}(\mathbb{K})$-models

Theorem 7

Let L *be an axiomatic extension of* $\mathrm{FL_{ew}}$ *and* $\mathbb{K}$ *a class of* L*-algebras.* *Let* LSM *be the expansion of* L *with the additional rules:*

$$(E) \qquad \frac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi} \qquad \frac{\varphi \leftrightarrow \psi}{\Diamond\varphi \leftrightarrow \Diamond\psi}$$

If L *is (finitely) strongly complete with respect to* $\mathbb{K}$*, then for each (finite)* $\Gamma \cup \{\varphi\} \subseteq Fm_{\Box\Diamond}$ *we have:*

$$\Gamma \vdash_{\mathrm{LSM}} \varphi \qquad \textit{iff} \qquad \Gamma \models_{\mathrm{SM}(\mathbb{K})} \varphi.$$

An axiomatization of the global logic of $\mathrm{SM}(\mathbb{K})$-models —in a general framework

Theorem 8

Let L *be a finitary protoalgebraic logic in a countable language* $\mathcal{L}$ *and* $\mathbb{K} \subseteq \mathbf{MOD}^*(\mathrm{L})$. *Let* LSM *be the expansion of* L *with the rules:*

$$(E) \qquad \frac{\varphi \Leftrightarrow \psi}{\Box\varphi \Leftrightarrow \Box\psi} \qquad \frac{\varphi \Leftrightarrow \psi}{\Diamond\varphi \Leftrightarrow \Diamond\psi}$$

If L *is strongly complete with respect to* $\mathbb{K}$,
then for each $\Gamma \cup \{\varphi\} \subseteq Fm_{\Box\Diamond}$ *we have:*

$$\Gamma \vdash_{\mathrm{LSM}} \varphi \qquad \textit{iff} \qquad \Gamma \models_{\mathrm{SM}(\mathbb{K})} \varphi.$$

An axiomatization of the global logic of $\mathrm{SM}(\mathbb{K})$-models —in a general framework

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Let L *be a finitary protoalgebraic logic in a countable language* $\mathcal{L}$ *and* $\mathbb{K} \subseteq \mathbf{MOD}^*(\mathrm{L})$. *Let* LSM *be the expansion of* L *with the rules:*

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If L *is finitely strongly complete with respect to* $\mathbb{K}$ *and* $\mathcal{L}$ *is finite, then for each finite* $\Gamma \cup \{\varphi\} \subseteq Fm_{\Box\Diamond}$ *we have:*

$$\Gamma \vdash_{\mathrm{LSM}} \varphi \qquad \textit{iff} \qquad \Gamma \models_{\mathrm{SM}(\mathbb{K})} \varphi.$$