Residuated lattices and twist-products

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based on a joint work with R. Cignoli

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Twist-structures

- J. Kalman, *Lattices with involution*, Trans. Amer. Math. Soc. **87** (1958), 485–491.
- M. Kracht, *On extensions of intermediate logics by strong negation*, J. Philos. Log. **27** (1998), 49–73.

Given a lattice $\mathbf{L} = \langle L, \vee, \wedge \rangle$ the twist constructions are obtained by considering

$$\mathbf{L}^{twist} = \langle L \times L, \sqcup, \sqcap, \sim \rangle$$

with the operations $\sqcup, \sqcap$ given by

$$(a, b) \sqcup (c, d) = (a \vee c, b \wedge d) \quad (1)$$

$$(a, b) \sqcap (c, d) = (a \wedge c, b \vee d) \quad (2)$$

$$\sim (a, b) = (b, a) \quad (3)$$

The operation $\sim$ satisfies:

1. $\sim\sim x = x$
2. $\sim (x \sqcap y) = \sim x \sqcup \sim y$
3. $\sim (x \sqcup y) = \sim x \sqcap \sim y$

When the lattice $\mathbf{L}$ has some additional operations, the construction $\mathbf{L}^{twist}$ can also be endowed with some additional operations.

This construction has been used to represent some well-known algebras:

- Nelson algebras

  Fidel, Vakarelov,
  Sendlewski,Cignoli, . . .

- Involutive residuated lattices

  Tsinakis, Wille
  Galatos, Raftery, . . .

- N4-lattices

  Odintsov

- Bilattices

  Ginsberg, Fitting,
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such that

- $\langle L, \cdot, e \rangle$ is a commutative monoid;
- $\langle L, \vee, \wedge \rangle$ is a lattice;
- $(\cdot, \rightarrow)$ is a residuated pair:

$$x \leq y \rightarrow z \quad \text{iff} \quad x \cdot y \leq z.$$

An *involution* on **L** is a unary operation $\sim$ satisfying the equations

$$\sim\sim x = x$$

and

$$x \rightarrow \sim y = y \rightarrow \sim x.$$

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$$\sim\sim x = x$$

and

$$x \to \sim y = y \to \sim x.$$

If $f := \sim e$, then $\sim x = x \to f$ and f satisfies the equation

$$(x \to f) \to f = x. \tag{4}$$

The element f is called a *dualizing element*.

Conversely, if $f \in L$ is a dualizing element and we define $\sim x = x \to f$ for all $x \in L$, then $\sim$ is an involution on $\mathbf{L}$ and $\sim e = f$.

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Therefore involutive residuated lattices are of the form:

$$\mathbf{L} = \langle L, \vee, \wedge, \cdot, \to, e, \sim \rangle$$

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We will deal with

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with e a dualizing element or equivalent $\sim x = x \to e$ an involution.

e-lattices.

By an e-lattice we mean a commutative residuated lattice **A** which satisfies the equation:

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The involution $\sim$ given by the prescription $\sim x = x \to e$ for all $x \in A$, satisfies the following properties:

M1 $\sim\sim x = x$,

M2 $\sim (x \vee y) = \sim x \wedge \sim y$,

M3 $\sim (x \wedge y) = \sim x \vee \sim y$,

M4 $\sim (x \cdot y) = x \to \sim y$,

M5 $\sim e = e$.

Lattice-ordered abelian groups with

$$x \cdot y = x + y,$$

$$x \to y = y - x$$

and $e = 0$ are examples of e-lattices.

Let $\mathbf{L} = \langle L, \vee, \wedge, \cdot, \rightarrow, e \rangle$ be an integral commutative residuated lattice.

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$$\mathbf{K}(\mathbf{L}) = \langle L \times L, \sqcup, \sqcap, \cdot_{K(L)}, \to_{K(L)}, (e, e)\rangle$$

with the operations $\sqcup, \sqcap, \cdot, \to$ given by

$$(a, b) \sqcup (c, d) = (a \vee c, b \wedge d) \tag{6}$$

$$(a, b) \sqcap (c, d) = (a \wedge c, b \vee d) \tag{7}$$

$$(a, b) \cdot_{K(L)} (c, d) = (a \cdot c, (a \to d) \wedge (c \to b)) \tag{8}$$

$$(a, b) \to_{K(L)} (c, d) = ((a \to c) \wedge (d \to b), a \cdot d) \tag{9}$$

The involution in pairs is given by

$$\sim (a, b) = (a, b) \rightarrow_{K(L)} (e, e) = (b, a). \tag{10}$$

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K(**L**) is an e-lattice.

Definition

We call $\mathbf{K}(\mathbf{L})$ the *full twist-product* obtained from $\mathbf{L}$, and every subalgebra $\mathbf{A}$ of $\mathbf{K}(\mathbf{L})$ containing the set $\{(a, e) : a \in L\}$ is called *twist-product* obtained from $\mathbf{L}$.

Recall that given a commutative residuated lattice $\mathbf{A} = (A, \vee, \wedge, \cdot, \to, e)$ its negative cone is given by

$$A^{-} = \{x \in A : x \leq e\}$$

and if we define

$$x \to_e y = (x \to y) \wedge e$$

then $\langle A^{-}, \vee, \wedge, \cdot, \to_e, e \rangle$ is an integral commutative residuated lattice.

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We aim to characterize the e-lattices that can be represented as twist-products obtained from their negative cones; i.e., If **A** is an *e*-lattice....

when does it happen that **A** is isomorphic to a subalgebra of $\mathbf{K}(\mathbf{A}^-)$?

Definition

We say that a commutative residuated lattice $\mathbf{L} = (L, \vee, \wedge, \cdot, \rightarrow, e)$ satisfies *distributivity at e* if the distributive laws

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \tag{11}$$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \tag{12}$$

hold whenever any of x, y, z is replaced by e.

Example: **L** is distributive at e, then it satisfies

$$e \vee (y \wedge z) = (e \vee y) \wedge (e \vee z) \quad (13)$$

$$x \wedge (e \vee z) = (x \wedge e) \vee (x \wedge z) \quad (14)$$

A K-lattice is an *e*-lattice satisfying distributivity at *e* and

$$(x \cdot y) \wedge e = (x \wedge e) \cdot (y \wedge e) \tag{15}$$

$$((x \wedge e) \to y) \wedge ((\sim y \wedge e) \to \sim x) = x \to y, \tag{16}$$

For every integral commutative residuated lattice **L** the twist-products **K**(**L**) are K-lattices.

It follows from the definition that K-lattices form a variety that we denote by $\mathbb{K}$.

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Lattice-ordered abelian groups are *e*-lattices that are not K-lattices.

It is well known and easy to verify that distributivity at e implies the quasiequation:

$$x \wedge e = y \wedge e \text{ and } x \vee e = y \vee e \text{ imply } x = y. \tag{17}$$

This is equivalent to:

$$\text{if } x \wedge e = y \wedge e \text{ and } \sim x \wedge e = \sim y \wedge e, \text{ then } x = y. \tag{18}$$

Theorem

Let $\mathbf{A}$ *be a K-lattice. The map* $\phi_{\mathbf{A}} : \mathbf{A} \to \mathbf{K}(\mathbf{A}^-)$ *given by*

$$x \mapsto (x \wedge e, \sim x \wedge e)$$

is an injective homomorphism.

$\phi_{\mathbf{A}}$ is a homomorphism.

- The preservation of the lattice operations relies on $\sim(x \vee y) = \sim x \wedge \sim y$ and distributivity at e. For $x, y \in A$

$$\phi_{\mathbf{A}}(x \wedge y) = ((x \wedge y) \wedge e, \sim(x \wedge y) \wedge e) =$$

$$((x \wedge e) \wedge (y \wedge e), (\sim x \vee \sim y) \wedge e) =$$

$$((x \wedge e) \wedge (y \wedge e), (\sim x \wedge e) \vee (\sim y \wedge e)) =$$

$$(x \wedge e, \sim x \wedge e) \sqcap (y \wedge e, \sim y \wedge e) = \phi_{\mathbf{A}}(x) \sqcap \phi_{\mathbf{A}}(y).$$

With similar ideas one can prove that $\phi_{\mathbf{A}}$ preserves the supremum.

Observe that

$$\phi_{\mathbf{A}}(\sim x) = (\sim x \wedge e, \sim\sim x \wedge e) = (\sim x \wedge e, x \wedge e) = \sim (x \wedge e, \sim x \wedge e).$$

Due to $\sim (x \cdot y) = \ x \to \sim y$, it is only left to check that $\phi_{\mathbf{A}}$ preserves $\cdot$.

Notice that

$$\phi_{\mathbf{A}}(x \cdot y) = ((x \cdot y) \wedge e, \sim (x \cdot y) \wedge e),$$

that can be rewritten as

$$((x \wedge e) \cdot (y \wedge e), (x \to\sim y) \wedge e). \tag{19}$$

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On the other hand,

$$\phi_{\mathbf{A}}(x) \cdot \phi_{\mathbf{A}}(y) =$$

$$((x \wedge e) \cdot (y \wedge e), ((x \wedge e) \to_e (\sim y \wedge e)) \wedge ((y \wedge e) \to_e (\sim x \wedge e))). \tag{20}$$

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To see that $\phi_{\mathbf{A}}(x \cdot y) = \phi_{\mathbf{A}}(x) \cdot \phi_{\mathbf{A}}(y)$ it remains to prove that the second components coincide.

We have

$$((x \wedge e) \to_e (\sim y \wedge e)) \wedge ((y \wedge e) \to_e (\sim x \wedge e)) =$$

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$$((x \wedge e) \to (\sim y)) \wedge e \wedge ((y \wedge e) \to (\sim x)) =$$

$$(x \to\sim y) \wedge e.$$

Finally, the injectivity of $\phi_{\mathbf{A}}$ follows at once from

$$x \wedge e = y \wedge e \text{ and } \sim x \wedge e = \sim y \wedge e \text{ imply } x = y.$$

So $\phi_{\mathbf{A}} : \mathbf{A} \to \mathbf{K}(\mathbf{A}^{-})$ given by

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Since for each $a \in A^{-}$,

$$\phi_{\mathbf{A}}(a) = (a, e),$$

it follows that by restriction, $\phi_{\mathbf{A}}$ defines an isomorphism from

$$\mathbf{A}^{-} \to \phi_{\mathbf{A}}(\mathbf{A})^{-}$$

.

Theorem

Every K-lattice **A** *is isomorphic to a twist-product obtained from its negative cone.*

Categories

The application

$$\mathbf{L} \mapsto \mathbf{K}(\mathbf{L})$$

and

$$f \mapsto (f, f)$$

from

$$\mathbb{ICRL} \to \text{K-lattices}$$

defines a functor.

The application

$$\mathbf{A} \mapsto \mathbf{A}^-$$

and

$$f \mapsto f\restriction_{\mathbf{A}^-}$$

from

$$\text{K-lattices} \to \mathbb{ICRL}$$

is also a functor which is left adjoint to the first.

Categories

Let $\mathcal{T}$ be the full subcategory of K-lattices whose objects are the total K-lattices, i.e.,

$$\mathbf{A} \cong \mathbf{K}(\mathbf{A}^{-})$$

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Theorem

The categories of integral commutative residuated lattices and $\mathcal{T}$ are equivalent categories.

Given a K-lattice $\mathbf{A}$ isomorphic to a subalgebra of $\mathbf{K}(\mathbf{A}^-)$,

how can we use information of the negative cone $\mathbf{A}^-$ to deduce some properties of $\mathbf{A}$?

Congruences

A first general result (not only for K-lattices) is that

The lattices $Cong(\mathbf{A})$ and $Cong(\mathbf{A}^{-})$ are isomorphic.

Translating equations

A K-lattice satisfies a lattice identity τ if and only if its negative cone satisfies τ and τ^d. In particular, a K-lattice is distributive if and only if its negative cone is distributive.

Representable K-lattices

A residuated lattice is *representable* if it is a subdirect product of linearly ordered residuated lattices. Given a subvariety $\mathbb{V} \subseteq \mathbb{CRL}$, the representable residuated lattices in $\mathbb{V}$ form a subvariety of $\mathbb{V}$ characterized by the equations

$$e \wedge (x \vee y) = (e \wedge x) \vee (e \wedge y) \tag{21}$$

and

$$e \wedge ((x \to y) \vee (y \to x)) = e. \tag{22}$$

Representable K-lattices

We introduce the following K-lattices:

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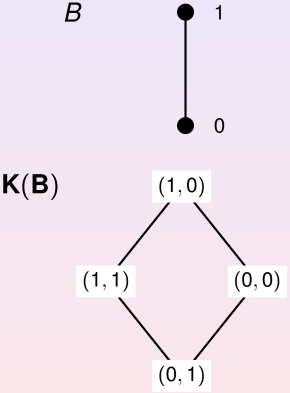

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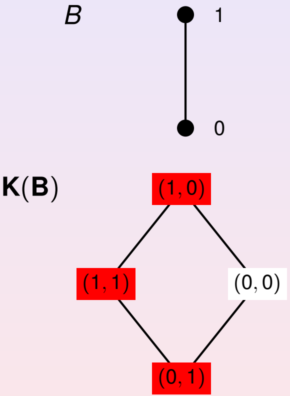
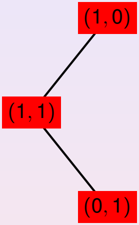


$\mathbf{P}_3$



1. Every three-element K-lattice is isomorphic to $\mathbf{P}_3$.
2. $\mathbf{P}_3$ is the only nontrivial K-lattice in which every element is comparable with e.
3. The K-lattice $\mathbf{P}_3$ is the only nontrivial totally ordered K-lattice.

Twist-products

For each integral commutative residuated lattice $\mathbf{L}$ we have a family of twist-products

$$\mathcal{K}_{\mathbf{L}} = \{\mathbf{S} \subseteq \mathbf{K}(\mathbf{L}) : \text{ for all } x \in L, (x, e) \in S\}.$$

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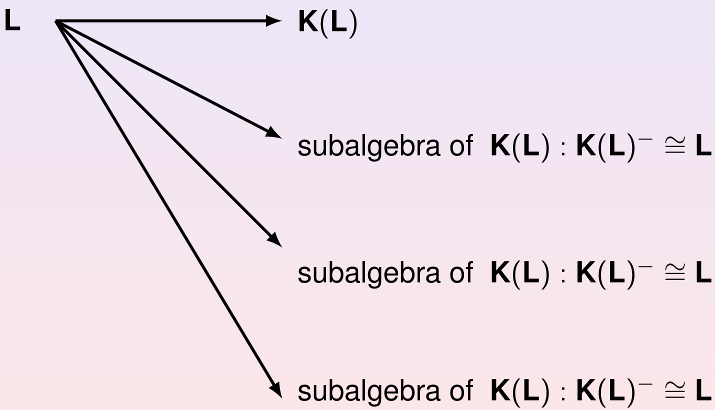
We aim to classify these twist-products.

- S. P. Odintsov, *Algebraic semantics for paraconsistent Nelson's logic*, J. Log. Comput. **13** (2003), 453–468.
- S. P. Odintsov, *On the representation of N4-lattices*, Stud. Log. **76** (2004), 385–405.
- S. P. Odintsov, *Constructive Negations and Paraconsistency*, Trends in Logic–Studia Logica Library 26. Springer. Dordrecht (2008)


$\mathbf{L}$
$\mathbf{K}(\mathbf{L})$
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$$(\mathbf{L}, F_1) \longrightarrow \mathbf{K}(\mathbf{L})$$

$$(\mathbf{L}, F_2) \longrightarrow \text{subalgebra of } \mathbf{K}(\mathbf{L}) : \mathbf{K}(\mathbf{L})^- \cong \mathbf{L}$$

$$(\mathbf{L}, F_3) \longrightarrow \text{subalgebra of } \mathbf{K}(\mathbf{L}) : \mathbf{K}(\mathbf{L})^- \cong \mathbf{L}$$

$$\vdots \qquad\qquad \vdots$$

$$(\mathbf{L}, F_n) \longrightarrow \text{subalgebra of } \mathbf{K}(\mathbf{L}) : \mathbf{K}(\mathbf{L})^- \cong \mathbf{L}$$

The finite MV-chain $\mathbf{L}_3$ given by

$$L_3 = \left\{0, \frac{1}{2}, 1\right\}$$

with the operations given by

$$x \cdot y = \max\{0, x + y - 1\} \qquad x \to y = \min\{1, 1 - x + y\}.$$

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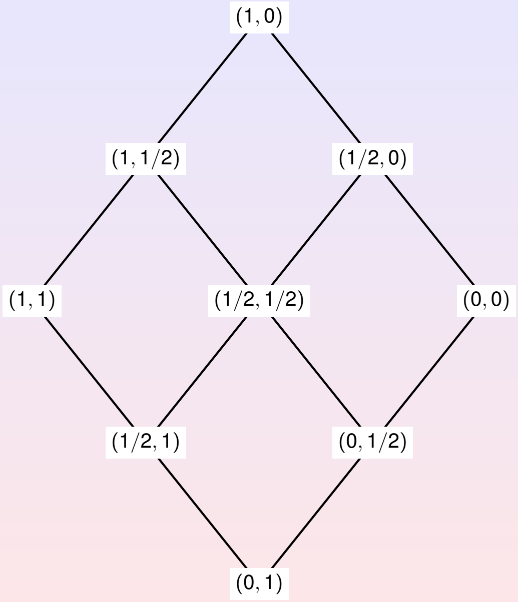
Recall that $\neg x = x \to 0$. One can always define $x \oplus y = \neg(\neg x \cdot \neg y)$ which is

$$x \oplus y = \min(0, x + y).$$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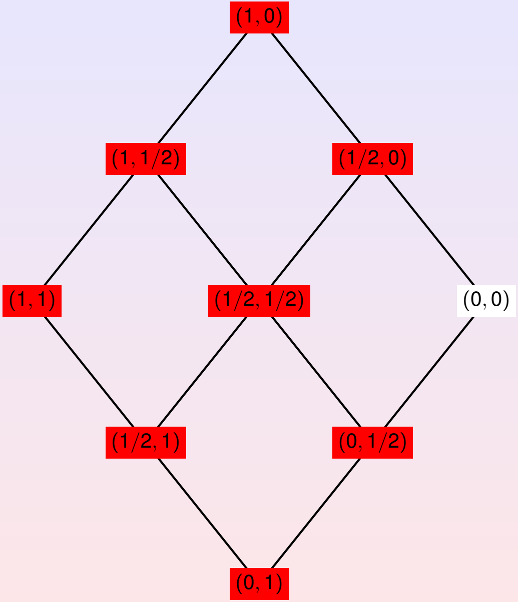
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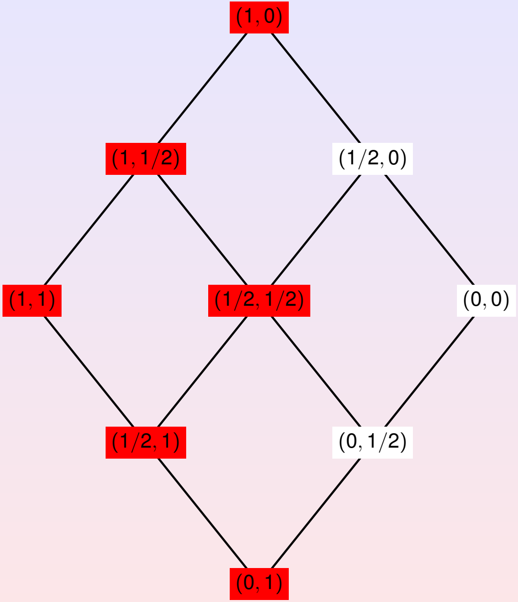
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$\mathcal{K}_{\mathbf{L}_3}$.

$$S_0 = \mathbf{K}(\mathbf{L}_3)$$

$$S_1 = \{(x, y) \in L_3 \times L_3 : x \oplus y = 1\}$$

$$S_{\frac{1}{2}} = \{(x, y) \in L_3 \times L_3 : x \oplus y \geq \frac{1}{2}\}.$$

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If we consider the three lattice filters of $\mathbf{L}_3$

$$F_1 = \{1\}, \quad F_{\frac{1}{2}} = \{1, \frac{1}{2}\}, \quad F_0 = \{1, \frac{1}{2}, 0\}$$

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$$S_0 = \mathbf{K} = \{(x, y) : x \oplus y \in F_0\}$$

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$$S_0 = \mathbf{K} = \{(x,y) : \neg x \to \neg\neg y \in F_0\}$$

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By an integral bounded commutative residuated lattice we mean an algebra

$$\mathbf{B} = \langle B, \vee, \wedge, \cdot, \rightarrow, e, 0\rangle$$

such that $\langle B, \vee, \wedge, \cdot, \rightarrow, e\rangle$ is an integral commutative residuated lattice and 0 is the lower bound of the lattice structure.

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such that $\langle B, \vee, \wedge, \cdot, \rightarrow, e\rangle$ is an integral commutative residuated lattice and 0 is the lower bound of the lattice structure.

Given an integral bounded commutative residuated lattice $\mathbf{B}$ we can define a negation on B as

$$\neg x = x \rightarrow 0.$$

By a *Glivenko residuated lattice* we mean an integral bounded commutative residuated lattice satisfying any of the equivalent conditions:

- $\neg\neg(\neg\neg x \to x) = e$.
- $\neg\neg(x \to y) = x \to \neg\neg y$.

Examples of Glivenko residuated lattices

- Integral involutive residuated lattices are trivially Glivenko.
- Heyting algebras are Glivenko.
- Integral bounded commutative residuated lattices that satisfy the hoop equation

$$x \wedge y = x \cdot (x \to y)$$

are Glivenko.

Let **B** be a Glivenko residuated lattice: there is a bijective correspondence between

regular lattice filters of B $\rightarrow$ admissible subalgebras of $\mathbf{K}(\mathbf{B})$

given by

$$F \mapsto \{(x, y) \in K(\mathbf{B}) : \neg x \to \neg\neg y \in F\}$$

whose inverse map is given by

$$S \mapsto \{x \in B : (0, x) \in S\}.$$

Conclusions

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2. We have studied representable K-lattices.
3. We have established a bijective correspondence among pairs of Glivenko residuated lattices and regular lattices filters and twist-products:

$$(L, F) \mapsto (S \subseteq K(L)).$$

Open problems

We believe that the key to understand K-lattices is the study of twist-products obtained from an arbitrary commutative integral residuated lattice **L**. This is equivalent to the investigation of admissible subalgebras of **K**(**L**).

1. Characterize admissible subalgebras of the full twist-product **K**(**B**) for **B** an arbitrary bounded integral commutative residuated lattice.
2. Characterize admissible subalgebras of the full twist-product **K**(**L**) for **L** an arbitrary integral commutative residuated lattice.

Thank you!

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