On Paraconsistent Weak Kleene Logic and Involutive Bisemilattices

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(Joint work with J. Gil-Férez, L. Peruzzi, and F. Paoli)

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Outline

1. Paraconsistent Weak Kleene Logic
2. Involutive bisemilattices
3. AAL approach to Paraconsistent Weak Kleene

Paraconsistent Week Kleene: Introduction

- The language: $\wedge, \vee, \neg, 0, 1$

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$\wedge$	0	$\frac{1}{2}$	1
0	0	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
1	0	$\frac{1}{2}$	1

$\vee$	0	$\frac{1}{2}$	1
0	0	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
1	1	$\frac{1}{2}$	1

$\neg$	
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

A closer look to $\mathbf{WK}$

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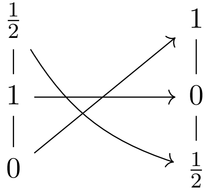

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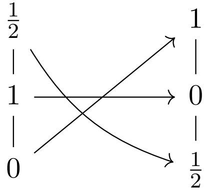

$$a \leqslant b \iff \neg b \leq \neg a$$

Counterexample to absorption:

$$1 \wedge (1 \vee \frac{1}{2}) = \frac{1}{2} \neq 1$$

Paraconsistent Weak Kleene: the logic

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- Hilbert system: any set of axioms for Classical Logic and

$$\text{[RMP]} \ \frac{\alpha \qquad \alpha \to \beta}{\beta} \quad \text{provided that } \mathrm{var}(\alpha) \subseteq \mathrm{var}(\beta)$$

Involutive bisemilattices

Definition

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We denote by $\mathcal{IBSL}$ the variety of involutive bisemilattices.

Examples

Every Boolean algebra, in particular the 2-element Boolean algebra $\mathbf{B}_2$, is an involutive bisemilattice.

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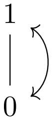

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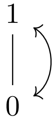
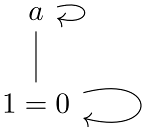
$\mathbf{B}_2 =$ 

$\mathbf{S}_2 =$ 

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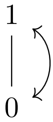

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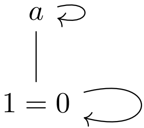

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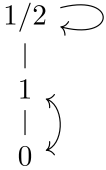

WK and $\mathcal{IBSL}$

Theorem

The only nontrivial subdirectly irreducible bisemilattices are $\mathbf{WK}$, $\mathbf{S}_2$, *and* $\mathbf{B}_2$, *up to isomorphism.*

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The only nontrivial subdirectly irreducible bisemilattices are $\mathbf{WK}$*,* $\mathbf{S}_2$*, and* $\mathbf{B}_2$*, up to isomorphism.*

Corollary

$\mathbb{V}(\mathbf{WK}) = \mathcal{IBSL}$.

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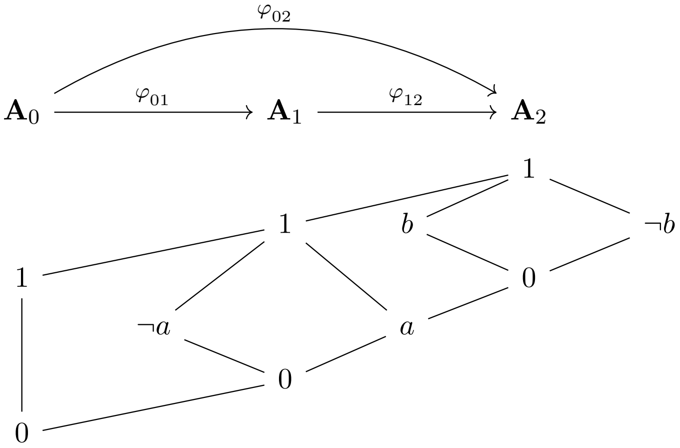
- if $g \in \nu$ is a constant, then $g^{\mathbf{T}} = g^{\mathbf{A}_{i_0}}$.

Płonka sums: example



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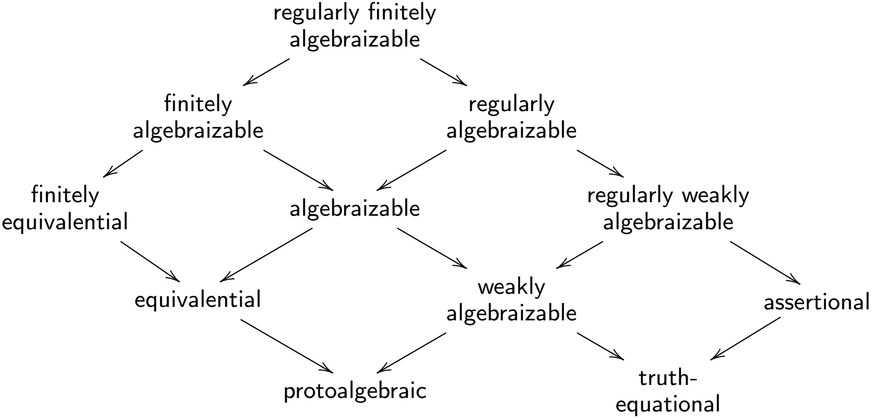
Corollary

$\mathcal{IBSL}$ is the variety satisfying exactly the regular identities satisfied by $\mathcal{BA}$.

Leibniz Hierarchy



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Theorem (Iso Thm)

If L is an algebraizable logic with equivalent algebraic semantics $\mathcal{K}$, then for every $\mathbf{A} \in \mathcal{K}$,

$$\boldsymbol{\Omega}^{\mathbf{A}} : \mathcal{F}\mathrm{i}_{\mathrm{L}}\, \mathbf{A} \to \mathrm{Co}_{\mathcal{K}}\, \mathbf{A}.$$

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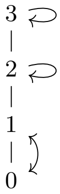
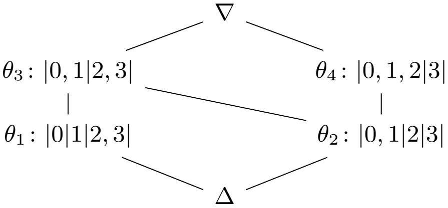
$\mathbf{C} =$ 



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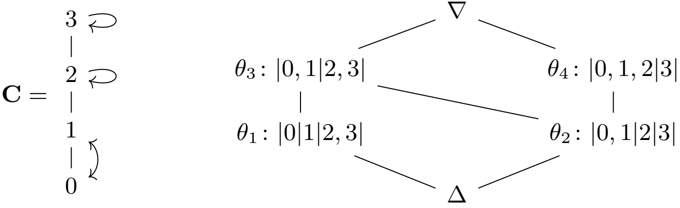

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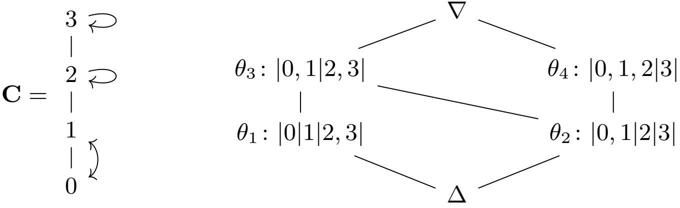

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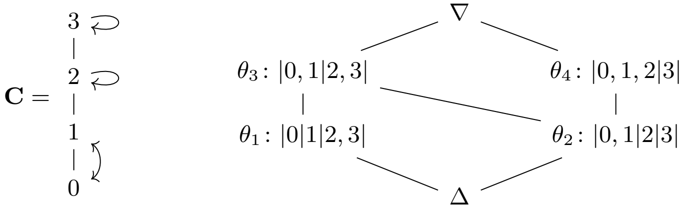

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- It follows that $\emptyset$ is also an L-filter, L is purely inferential, and this leads to a contradiction.

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- Thus, $v[\{p\} \cup p \Rightarrow q] = \{1/2\}$, while $v(q) = 0$, which is a contradiction.

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  - $v(\neg(\neg p \vee p)) = 1/2$.
  - $v(\neg(\neg q \vee q)) = 0$.
- Therefore $\neg(\neg p \vee p) \dashv\vDash_{\mathrm{PWK}} \neg(\neg q \vee q)$, does not hold. That is, $\dashv\vDash_{\mathrm{PWK}}$ is not a congruence.

AAL

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$$\mathsf{Alg}^*(\mathrm{L}) = \{\mathbf{A} : \text{there is a reduced model } \langle \mathbf{A}, F \rangle \text{ of } \mathrm{L}\}.$$

The Leibniz congruence

Lemma

If $\mathbf{A}$ *is an algebra of type of* $\mathcal{IBSL}$ *and* $F \in \mathcal{F}\mathrm{i}_{\mathrm{PWK}}\,\mathbf{A}$*, then for every* $a, b \in A$*,* $\langle a, b\rangle \in \boldsymbol{\Omega}^{\mathbf{A}} F$ *if and only if for every* $c \in A$*,*

$$a \vee c \in F \iff b \vee c \in F \quad \text{and} \quad \neg a \vee c \in F \iff \neg b \vee c \in F.$$

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Theorem

$\mathsf{Alg}^*(\mathrm{PWK}) \subseteq \mathcal{IBSL}$.

Let $\mathbf{B} \in \mathcal{IBSL}$, $b \in B$ is positive iff $1 \leqslant b$.

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Moreover, $\langle \mathbf{B}, F \rangle \in \mathsf{Mod}^*(\mathrm{PWK})$ *if and only if* $\mathbf{B}$ *is an involutive bisemilattice satisfying the above condition and* $F = P(\mathbf{B})$*, the set of positive elements, which is given by:*

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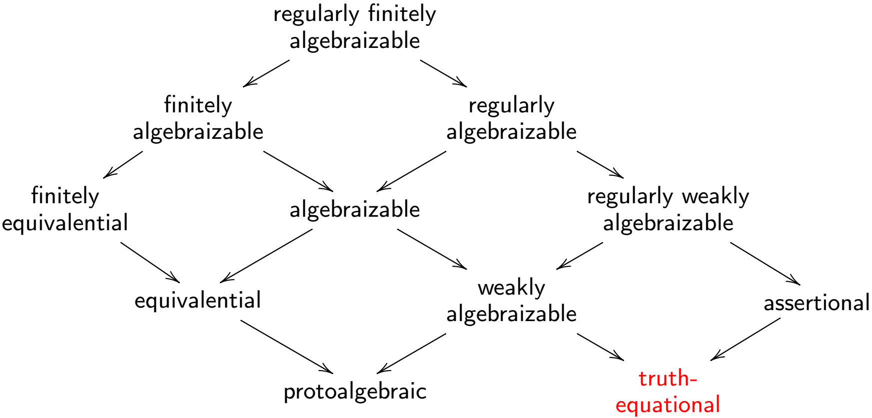
Corollary

PWK *is truth-equational.*

PWK in the Leibniz Hierarchy



Work in progress

- Natural duality for involutive bisemilattices (joint with A. Loi and L. Peruzzi)

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- Sequent calculi for PWK and Gentzen algebraizability (joint with M. Pra Baldi)

Thank you!