IUML-Algebras of Refinements of Orthopairs

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work with
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Objective

Establishing a relationship between finite IUML-algebras and sequences of successive refinements of orthopairs, using Sobociński conjunction.

IUML-algebras

An idempotent uninorm mingle logic algebra (*IUML-algebra*) is an idempotent commutative bounded distributive residuated lattice

$$(A, \wedge, \vee, *, \to, \bot, \top, e)$$

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Example

The structure $([0,1], \wedge, \vee, *, \to, 0, 1, 1/2)$ is an IUML-algebra.

$$x * y = \begin{cases} \min(x, y) & \text{if } x \leq 1 - y \\ \max(x, y) & \text{otherwise.} \end{cases} \qquad x \to y = \begin{cases} \max(1 - x, y) & \text{if } x \leq y \\ \min(1 - x, y) & \text{otherwise.} \end{cases}$$

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Sobociński conjunction and implication:

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0	0	0	0
1/2	0	1/2	1
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$\to$	0	1/2	1
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Remark

$(\{0, 1/2, 1\}, \wedge, \vee, *, \rightarrow, 0, 1, 1/2)$ generates every three-valued IUML-algebra.

IUML-algebras and Forests

Given F a finite forest, we set

$$SP(F) = \{(X^1, X^2) : X^1 \text{and } X^2 \text{are disjoint upsets of } F\}.$$

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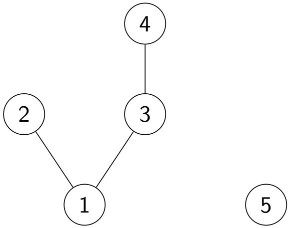
   where

- $X \diamond Y = \uparrow((X^0 \cap Y^1) \cup (Y^0 \cap X^1))$;
- $X^0 = F \setminus (X^1 \cup X^2)$ and $Y^0 = F \setminus (Y^1 \cup Y^2)$.

Operations on $SP(F)$: $*$

Example



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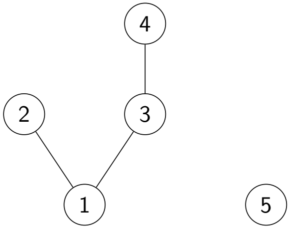

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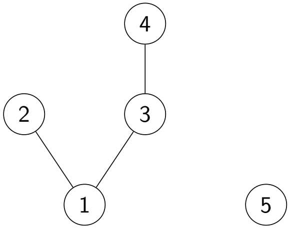

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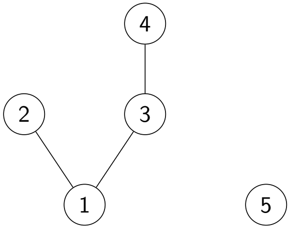

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$$(\emptyset \cup \{3,4\}, \{2,4,5\}/\{3,4\}) = (\{3,4\},\{2,5\}).$$

Forests and IUML-algebras

Theorem

For every finite forest F, $(SP(F), \sqcap, \sqcup, *, \to, (\emptyset, F), (F, \emptyset), (\emptyset, \emptyset))$ is an IUML-algebra. Vice-versa, each finite IUML-algebra is isomorphic with $SP(F)$ for some finite forest F.

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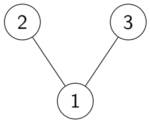

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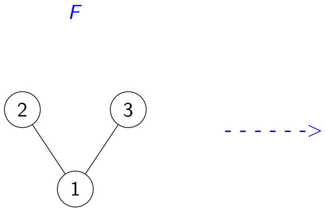

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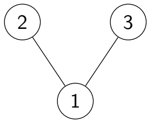

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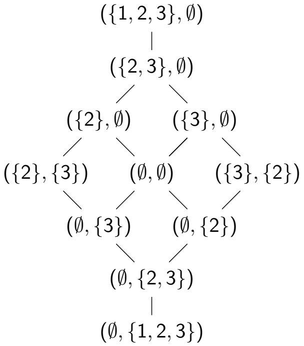


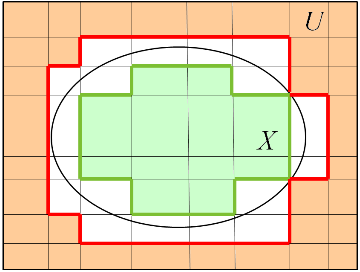
Orthopairs

Let P be a partition of U and $X \subseteq U$. The orthopair of X determined by P is

$$(\mathcal{L}_P(X), \mathcal{E}_{\mathcal{P}}(X))$$

where

- lower approximation
  $\mathcal{L}_P(X) = \{x \in N : N \in P \text{ and } N \subseteq X\}$,
- impossibility domain
  $\mathcal{E}_P(X) = \{x \in N : N \in P \text{ and } N \cap X = \emptyset\}$.



Example

Let $U = \{a, b, c, d, e, f\}$, $P = \{\{a, b\}, \{c, d, e, f\}\}$ and $X = \{a, b, c\}$. Then

$$(\mathcal{L}_P(\{a, b, c\}), \mathcal{E}_P(\{a, b, c\})) = (\{a, b\}, \emptyset)$$

Orthopairs and IUML-algebras

Theorem

Let P be a partition of U. We set

$$O_P = \{(\mathcal{L}_P(X), \mathcal{E}_P(X)) : X \subseteq U\}.$$

Then

$$(O_P, \wedge, \vee, *, \rightarrow, (U, \emptyset), (\emptyset, U), (\emptyset, \emptyset))$$

where

$$(A, B) * (C, D) = ((A \cap C) \cup (A \setminus (C \cup D)) \cup (C \setminus (A \cup B)), B \cup D),$$

is a 3-valued IUML-algebra.

Refinement sequences

A sequence $\mathcal{P} = P_0, \ldots, P_n$ of partial partitions of U is a refinement sequence of partial partitions of U if each element of P_i is contained in an element of P_{i-1}, for $i = 1, \ldots, n$.

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Example

Let $U = \{a, b, c, d, e, f, g, h\}$ and

- $P_0 = \{\{a, b, c, d\}, \{e, f, g, h\}\}$;
- $P_1 = \{\{a, b\}, \{c, d\}, \{e, f, g\}\}$.

(P_0, P_1) is a refinement sequence of U.

Sequence of Refinement of Orthopairs

Let $\mathcal{P} = P_0, \ldots, P_n$ be a refinement sequence of U. For every $X \subseteq U$, $\mathcal{P}$ determines the sequence of refinement of orthopairs

$$\mathcal{O}_{\mathcal{P}}(X) = ((\mathcal{L}_0(X), \mathcal{E}_0(X)), \ldots, (\mathcal{L}_n(X), \mathcal{E}_n(X))).$$

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Let $U = \{a, b, c, d, e, f, g, h\}$, $X = \{a, b, c\}$ and $\mathcal{P} = (P_0, P_1)$ such that

- $P_0 = \{\{a, b, c, d\}, \{e, f, g, h\}\}$ and
- $P_1 = \{\{a, b\}, \{c, d\}, \{e, f, g\}\}$.

Then

$$\mathcal{O}_{\mathcal{P}}(\{a, b, c\}) = ((\emptyset, \{e, f, g, h\}), (\{a, b\}, \{e, f, g\}))$$

Partial partitions and Forests

Let $\mathcal{P}$ be a refinement sequence of U.

$$\mathcal{P} \longrightarrow (F_{\mathcal{P}}, \leq_{\mathcal{P}})$$

where

1. $F_{\mathcal{P}} = \bigcup_{i=0}^{n} P_i$ and
2. $\leq_{\mathcal{P}}$ is the reverse inclusion.

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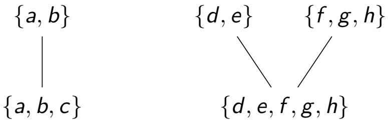

Let $\mathcal{P}$ be a refinement sequence of U and $X \subseteq U$. Then

- $X^1_{\mathcal{P}} = \{N \in F_{\mathcal{P}} : N \subseteq X\}$
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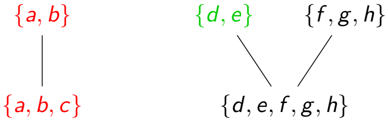

Then $(X^1_{\mathcal{P}}, X^2_{\mathcal{P}}) = (\{\{d, e\}\}, \{\{a, b, c\}, \{a, b\}\})$.

Results (1/5)

Theorem

Let $\mathcal{P}$ be a refinement sequence of U. Then the map

$$f : O_{\mathcal{P}}(X) \in \mathcal{O}_{\mathcal{P}} \mapsto (X_{\mathcal{P}}^1, X_{\mathcal{P}}^2) \in SO(F_{\mathcal{P}})$$

is bijective.

Results (2/5)

Theorem

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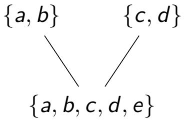
1. $SO(F_{\mathcal{P}}) \subseteq SP(F_{\mathcal{P}})$;
2. $SO(F_{\mathcal{P}}) = SP(F_{\mathcal{P}})$ *if and only if every node of* $F_{\mathcal{P}}$ *strictly contains the unions of its successors. Trivially,* $(SO(F_{\mathcal{P}}), \sqcap, \sqcup, *, \rightarrow, (\emptyset, F_{\mathcal{P}}), (F_{\mathcal{P}}, \emptyset), (\emptyset, \emptyset))$ *is an IUML-algebra.*

Example

$F_{\mathcal{P}}$



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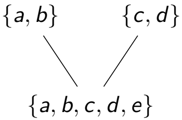

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Example

$(SO(F_{\mathcal{P}}), \sqcap, \sqcup, *, \rightarrow, (\emptyset, F_{\mathcal{P}}), (\emptyset, F_{\mathcal{P}}), (\emptyset, \emptyset))$

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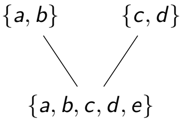

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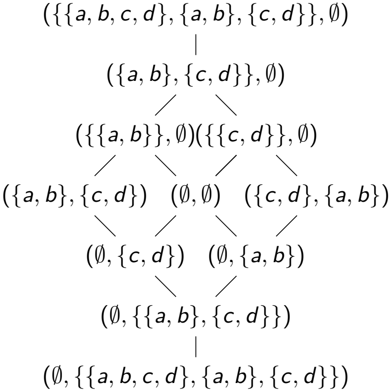

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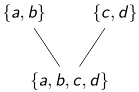

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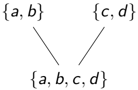
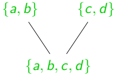
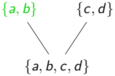


$SO(F_{\mathcal{P}})$



$(\{\{a,b,c,d\},\{a,b\},\{c,d\}\},\emptyset)$



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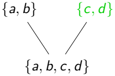

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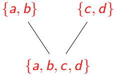
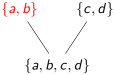


$(\emptyset,\{\{a,b,c,d\},\{a,b\},\{c,d\}\})$



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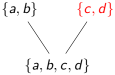

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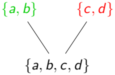

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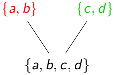
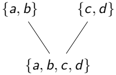
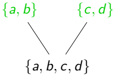


$(\{\{c,d\}\},\{\{a,b\}\})$

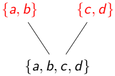


$(\emptyset,\emptyset)$

$SP(F_{\mathcal{P}}) \setminus SO(F_{\mathcal{P}})$



$(\{\{a,b\},\{c,d\}\},\emptyset)$



$(\emptyset,\{\{a,b\},\{c,d\}\})$

Results (3/5)

Theorem

Let $\mathcal{P}$ be a refinement sequence of U. Then there exists a refinement sequence $\mathcal{P}'$ of U such that the map

$$g : (X^1_{\mathcal{P}}, X^2_{\mathcal{P}}) \in SO(F_{\mathcal{P}}) \mapsto (X^1_{\mathcal{P}'}, X^2_{\mathcal{P}'}) \in SP(F_{\mathcal{P}'})$$

is bijective.

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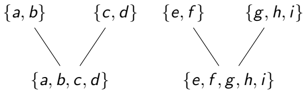

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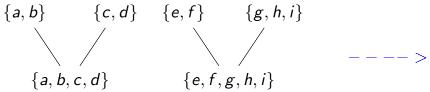

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Let $\mathcal{P}$ be a refinement sequence of U. Then there exists a refinement sequence $\mathcal{P}'$ of U such that the map

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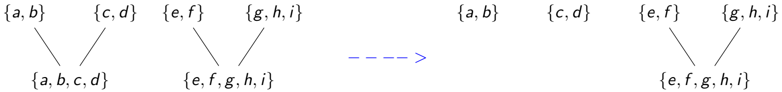
Example



Results (4/5)

Theorem

Let $\mathcal{P}$ be a refinement sequence of U and $g : SO(F_{\mathcal{P}}) \mapsto SP(F_{\mathcal{P}'})$. Then

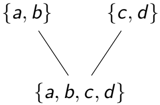
$$(SO(F_{\mathcal{P}}), \sqcap_g, \sqcup_g, *_g, \rightarrow_g, (\emptyset, F_{\mathcal{P}}), (\emptyset, F_{\mathcal{P}}), (\emptyset, \emptyset))$$

- $(X^1_{\mathcal{P}}, X^2_{\mathcal{P}}) \sqcap_g (Y^1_{\mathcal{P}}, Y^2_{\mathcal{P}}) := g^{-1}((X^1_{\mathcal{P}'}, X^2_{\mathcal{P}'}) \sqcap (Y^1_{\mathcal{P}'}, Y^2_{\mathcal{P}'}))$,
- $(X^1_{\mathcal{P}}, X^2_{\mathcal{P}}) \sqcup_g (Y^1_{\mathcal{P}}, Y^2_{\mathcal{P}}) := g^{-1}((X^1_{\mathcal{P}'}, X^2_{\mathcal{P}'}) \sqcup (Y^1_{\mathcal{P}'}, Y^2_{\mathcal{P}'}))$,
- $(X^1_{\mathcal{P}}, X^2_{\mathcal{P}}) *_g (Y^1_{\mathcal{P}}, Y^2_{\mathcal{P}}) := g^{-1}((X^1_{\mathcal{P}'}, X^2_{\mathcal{P}'}) * (Y^1_{\mathcal{P}'}, Y^2_{\mathcal{P}'}))$,
- $(X^1_{\mathcal{P}}, X^2_{\mathcal{P}}) \rightarrow_g (Y^1_{\mathcal{P}}, Y^2_{\mathcal{P}}) := g^{-1}((X^1_{\mathcal{P}'}, X^2_{\mathcal{P}'}) \rightarrow (Y^1_{\mathcal{P}'}, Y^2_{\mathcal{P}'}))$.

is an IUML-algebra.

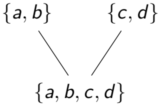
Example

$F_{\mathcal{P}}$



Example

$F_{\mathcal{P}}$



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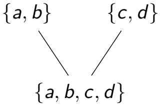
Example

$F_{\mathcal{P}}$



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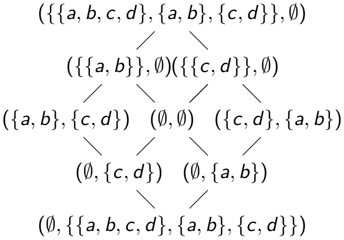
$(SO(F_{\mathcal{P}}), \sqcap_g, \sqcup_g, *_g, \rightarrow_g, (\emptyset, F_{\mathcal{P}}), (\emptyset, F_{\mathcal{P}}), (\emptyset, \emptyset))$



$\simeq SP(F_{\mathcal{P}'})$

Results (5/5)

Theorem

Let $\mathcal{P} = (P_0, \ldots, P_n)$ a refinement sequence of U, the structure of IUML-algebra on $SO(F_{\mathcal{P}})$ induces on sequences of orthopairs the following operation, for every $X, Y \subseteq U$:

$$\mathcal{O}_{\mathcal{P}}(X) \odot \mathcal{O}_{\mathcal{P}}(Y) = ((A_0, B_0), \ldots, (A_n, B_n))$$

where for each $i = 1, \ldots, n$, we firstly set

$$(A'_i, B'_i) = (\mathcal{L}_i(X), \mathcal{E}_i(X)) * (\mathcal{L}_i(Y), \mathcal{E}_i(Y))$$

and then $A_0 = A'_0$ and for $i > 0$:

$$A_{i+1} = A'_{i+1} \cup \{B \in P_{i+1} \mid B \subseteq A_i\}$$

while $B_i = B'_i \setminus A_i$.

Example

Given the refinement sequence $\mathcal{P} = (P_0, P_1, P_2)$:

- $P_0 = \{\{a, b, c, d, e, f, g, h, i\}\}$,
- $P_1 = \{\{a, b, c\}, \{d, e, f, g, h\}\}$,
- $P_2 = \{\{a, b\}, \{d, e\}, \{f, g\}\}$.

If $X = \{c, d, e, f, g, h\}$ and $Y = \{a, b, c, d, e\}$, then

$$O_{\mathcal{P}}(X) = \begin{pmatrix} (\emptyset, \emptyset) \\ (\{d, e, f, g, h\}, \emptyset) \\ (\{d, e, f, g\}, \{a, b\}) \end{pmatrix} \qquad O_{\mathcal{P}}(Y) = \begin{pmatrix} (\emptyset, \emptyset) \\ (\{a, b, c\}, \emptyset) \\ (\{a, b, d, e\}, \{f, g\}) \end{pmatrix}$$

$$O_{P_0}(X) * O_{P_0}(Y) = (\emptyset, \emptyset) * (\emptyset, \emptyset) = (\emptyset, \emptyset)$$
$$O_{P_1}(X) * O_{P_1}(Y) = (\{d, e, f, g, h\}, \emptyset) * (\{a, b, c\}, \emptyset) = (\{a, b, c, d, e, f, g, h\}, \emptyset)$$
$$O_{P_2}(X) * O_{P_2}(Y) = (\{d, e, f, g\}, \{a, b\}) * (\{a, b, d, e\}, \{f, g\}) = (\{d, e\}, \{a, b, f, g\})$$

$$(\emptyset, \emptyset) \longrightarrow (\emptyset, \emptyset)$$
$$(\{a, b, c, d, e, f, g, h\}, \emptyset) \longrightarrow (\{a, b, c, d, e, f, g, h\}, \emptyset)$$
$$(\{d, e\}, \{a, b, f, g\}) \longrightarrow (\{d, e\} \cup \{a, b\} \cup \{f, g\}, \{\{a, b, f, g\} \setminus \{a, b\}\} \setminus \{f, g\})$$
$$= (\{a, b, d, e, f, g\}, \emptyset)$$

Future Work

We plan

- to find a natural interpretation of the last operations;
- to generalize to other operations between orthopairs and to study the obtained algebraic structures.

Thanks for the attention