Logic of knowledge and belief for Sceptical Agents
(a substructural epistemic logic)

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An alternative modal view on knowledge

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- working with collections of data — those might be *incomplete* and *inconsistent*.
- The agent (e.g. by weighting the evidence) eventually accepts some *available data* as knowledge,
- but only *confirmed data* might be accepted (certified knowledge).

Logical formalism

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- Modal part consists of epistemic diamond operators of confirmed knowledge and confirmed belief.
- We start from a basic system, allowing for further modularity (distributive non-associative commutative Lambek calculus with a negation).

Results

- A concept of confirmed belief or knowledge can be modeled as a diamond modality over relational semantics for substructural logics.
- Strong completeness, canonicity, correspondence, FMP via filtration.
- Structural (display) proof theory, cut elimination.
- Common knowledge and common belief as fixed points, infinitary, strongly complete, proof systems.

The language

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$$\begin{aligned}\varphi ::= \quad & p \mid t \mid \varphi \otimes \varphi \mid \varphi \to \varphi \\ & \top \mid \bot \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \\ & \neg\varphi \mid \langle k \rangle \varphi \mid \langle b \rangle \varphi\end{aligned}$$

$\langle k \rangle$ is the confirmed knowledge operator,
$\langle b \rangle$ is the confirmed belief operator.

The frames

A *frame* $F = (X, \leq, R, L, C)$, where

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A *frame* $F = (X, \leq, R, L, C)$, where

- $(X, \leq)$ is a poset of information states,
- R is a ternary monotone relation on X:

$$Rxyz \wedge x' \leq x \wedge y' \leq y \wedge z' \geq z \longrightarrow Rx'y'z',$$

  satisfying $Rxyz \longrightarrow Ryxz$.
- L, the set of *logical states*, is a nonempty upwards closed subset of $(X, \leq)$, satisfying

$$x \leq y \ \text{ iff } \ (\exists z \in L)\ Rzxy \text{ iff } \ (\exists z \in L)\ Rxzy$$

- C is a binary *compatibility* monotone relation on X:

$$xCy \wedge x' \leq x \wedge y' \leq y \longrightarrow x'Cy',$$

  we consider C to be symetric.

Interpreting the propositional part

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A valuation is a map $V : \mathsf{Prop} \longrightarrow \mathcal{U}X$

- $x \Vdash p$ iff $x \in V(p)$
- $x \Vdash t$ iff $x \in L$
- $x \Vdash \top$ and $x \nVdash \bot$
- $x \Vdash \varphi \wedge \psi$ iff $x \Vdash \varphi$ and $x \Vdash \psi$
- $x \Vdash \varphi \vee \psi$ iff $x \Vdash \varphi$ or $x \Vdash \psi$
- $x \Vdash \neg\varphi$ iff for all y, xCy implies $y \nVdash \varphi$
- $x \Vdash \varphi \otimes \psi$ iff there are y, z, $Ryzx$ and $y \Vdash \varphi$ and $z \Vdash \psi$
- $x \Vdash \varphi \leftarrow \psi$ iff for all y, z, $Rxyz$ and $y \Vdash \varphi$ implies $z \Vdash \psi$
- $x \Vdash \varphi \rightarrow \psi$ iff for all y, z, $Ryxz$ and $y \Vdash \varphi$ implies $z \Vdash \psi$

Basic semantical notions

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- Frame validity: $F, V \Vdash \varphi$ iff $(\forall x \in L)\ x \Vdash \varphi$
- Local consequence: $\varphi \vDash_{F,V} \psi$ iff $(\forall x)\ x \Vdash \varphi$ implies $x \Vdash \psi$
- Valid implications: $\vDash \varphi \to \psi$ iff $(\forall F, V)\ \varphi \vDash_{F,V} \psi$
- A set $\{\varphi \mid x \Vdash \varphi\}$ is a prime theory.
  For all φ, $\{x \mid x \Vdash \varphi\}$ is upward closed.
- Dual connection between distributive non-associative FL_e algebras and the frames defined above.

Adding epistemic source relations

S^k and S^b are binary monotone relations on $(X, \leq)$:

$$x' \leq xS^k y \leq y' \quad \text{implies} \quad x' S^k y'$$
$$x' \leq xS^b y \leq y' \quad \text{implies} \quad x' S^b y',$$

satisfying (all or some of) the conditions:

$$sS^b x \text{ and } s' S^b x \quad \text{implies} \quad sCs' \tag{1}$$
$$sS^k x \quad \text{implies} \quad sS^b x \tag{2}$$
$$sS^k x \quad \text{implies} \quad s \leq x \tag{3}$$
$$sS^k x \quad \text{implies} \quad xCs \tag{4}$$

We read $sS^k x$ as s is a reliable source confirming knowledge in x. Similarly for belief.

Interpreting the modal part

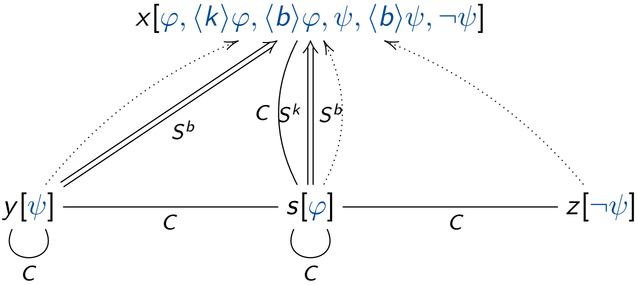
- $x \Vdash \langle k \rangle \varphi$ iff $\exists s\, (sS^k x \wedge s \Vdash \varphi)$ confirmed knowledge
- $x \Vdash \langle b \rangle \varphi$ iff $\exists s\, (sS^b x \wedge s \Vdash \varphi)$ confirmed belief

Properties of the source relations

- Sources of belief are mutually compatible,
- $S^k \subseteq S^b$ (as well as $S^k \subseteq \leq \cap C$) implies that sources for knowledge are mutually compatible (*do not contradict each other*).
- Sources are self-compatible (therefore *consistent*).
- $S^k \subseteq \leq$ implies that what is known is satisfied in the current state.

Knowledge implies belief, is consistent and factive. Belief is consistent.

Example



$x[\varphi, \langle k\rangle\varphi, \langle b\rangle\varphi, \psi, \langle b\rangle\psi, \neg\psi]$
S^b
C
S^k
S^b
$y[\psi]$
C
$s[\varphi]$
C
$z[\neg\psi]$
C
C

Axioms and corresponding classes of frames

Axiom or rule	condition
$\langle k\rangle\varphi \to \varphi$	$sS^kx \longrightarrow s \leq x$
$\langle k\rangle\varphi \to \langle b\rangle\varphi$	$sS^kx \longrightarrow sS^bx$
$\langle b\rangle\varphi \wedge \langle b\rangle\neg\varphi \to \bot$	$sS^bx \wedge s'S^bx \longrightarrow sCs'$
$\langle k\rangle\varphi \wedge \neg\varphi \to \bot$	$sS^kx \longrightarrow sCx$
$\langle k\rangle\varphi \to \langle k\rangle\langle k\rangle\varphi$	$sS^kx \longrightarrow \exists s'\ (sS^ks'S^kx)$
$\langle b\rangle\varphi \to \langle b\rangle\langle b\rangle\varphi$	$sS^bx \longrightarrow \exists s'\ (sS^bs'S^bx)$
$\langle b\rangle\varphi \to \langle b\rangle\langle k\rangle\varphi$	$sS^bx \longrightarrow \exists s'\ (sS^ks'S^bx)$
$\langle k\rangle\varphi \wedge \langle k\rangle\psi \to \langle k\rangle(\varphi \wedge \psi)$	$sS^kx \wedge tS^kx \longrightarrow \exists v\ (vS^kx \wedge s, t \leq v)$
$\vdash \varphi / \vdash \langle k\rangle\varphi$	$(\forall x \in L)(\exists s \in L)\ sS^kx$

Further modularity - negation

axiom	condition
$\neg\top \to \bot$	$(\forall x)(\exists y)xCy$
$\varphi \wedge \neg\varphi \to \bot$	xCx
$\top \to \varphi \vee \neg\varphi$	$xCy \to y \leq x$

Example: in presence of $\top \to \varphi \vee \neg\varphi$, the factivity scheme $\langle k\rangle\varphi \to \varphi$ is derivable from the stronger consistency scheme $\langle k\rangle\varphi \wedge \neg\varphi \to \bot$. In presence of $\varphi \wedge \neg\varphi \to \bot$, it is the other way round.

Intuitionistic epistemic logic

(i) Simplest way of arriving at intuitionistic epistemic logic is given by FL_{ewc} and its corresponding class of frames putting $Rxxx$ and $Rxyz \longrightarrow x \leq z$.

(ii) From the standard semantics of intuitionistic logic: for a poset $(X, \leq)$, put $L = X$, let S^k to be any monotone relation satisfying $S^k \subseteq \leq$, and define the remaining relations as follows:

$$\begin{array}{rcl} Rxyz & \text{iff} & x \leq z \text{ and } y \leq z \\ Cxy & \text{iff} & \exists z(x \leq z \text{ and } y \leq z) \end{array}$$

The modality is not trivial ($\varphi \nvdash \langle k \rangle \varphi$), and neither it commutes with the conjunction nor distributes to the implication.

(iii) Consider $(X, \leq)$ to be a rooted tree with the root r. Put $rS^k x$ for all $x \in X$ (the root r is a universal source).
In this class of frames, $\langle k \rangle$ commutes with conjunction, distributes to implication, positive introspection axiom becomes valid, as well as negative introspection axiom.

What about classical negation?

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Consider frames validating $\top \vdash \varphi \vee \neg\varphi$:

- The corresponding frame condition together with the symmetry of C entail that xCy implies $x = y$,
- together with the scheme $\varphi \wedge \neg\varphi \vdash \bot$ and the corresponding condition xCx we obtain that C is the equality.
- By the condition that $S^k \subseteq C$, also $xS^k y$ implies $x = y$ (self-sources only). The positive introspection axiom becomes valid, while the negative introspection may fail.
- By the weaker mutual compatibility condition $x, zS^k y$ implies $x = z$ (one source only). Introspection axioms may fail.

Box or Diamond?

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- Consider a monotone neighborhood model $(W, N : PW \longrightarrow PW)$, where $||\Box\alpha|| = N||\alpha||$ (knowledge)
- define a frame $(PW, \supseteq)$ with a relation:

$$xSy \ \equiv_{df} \ N(x) \supseteq y$$

- put $$x \Vdash p \ \Leftrightarrow \ x \subseteq ||p||$$
- Then $(\wedge, \vee, \neg, \Box)$ translates to $(\wedge, \sqcup, \neg, \langle k \rangle)$ where $\sqcup$ is the inquisitive disjunction, and $\neg$ is interpreted by

$$xCy \ \equiv_{df} x \not\subseteq \overline{y}$$

Axiomatization (dFLe + modal axioms)

$$\frac{\varphi \to \psi}{\langle k \rangle \varphi \to \langle k \rangle \psi} \qquad \langle k \rangle (\varphi \vee \psi) \to \langle k \rangle \varphi \vee \langle k \rangle \psi \qquad \langle k \rangle \bot \to \bot$$

$$\frac{\varphi \to \psi}{\langle b \rangle \varphi \to \langle b \rangle \psi} \qquad \langle b \rangle (\varphi \vee \psi) \to \langle b \rangle \varphi \vee \langle b \rangle \psi \qquad \langle b \rangle \bot \to \bot$$

$$\langle k \rangle \varphi \to \varphi \qquad \langle k \rangle \varphi \to \langle b \rangle \varphi \qquad \langle b \rangle \varphi \wedge \langle b \rangle \neg \varphi \to \bot$$

Additional modal axioms

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Strong consistency: $\langle k\rangle\varphi \wedge \neg\varphi \rightarrow \bot$

Stalnaker's axiom: $\langle b\rangle\varphi \rightarrow \langle b\rangle\langle k\rangle\varphi$

Belief introspection: $\langle b\rangle\varphi \rightarrow \langle b\rangle\langle b\rangle\varphi$

Knowledge introspection: $\langle k\rangle\varphi \rightarrow \langle k\rangle\langle k\rangle\varphi$

Completeness via canonical model

Theorem (Strong Completeness) The axiomatization (+ Ax) is strongly complete with respect to the class of corresponding epistemic frames.

$$\Gamma \nvdash \varphi \text{ implies } \Gamma \nvDash_{\mathcal{F}(Ax)} \varphi$$

Proof — the canonical model construction. Canonical states = prime theories ordered by inclusion, canonical relations defined as usual. All axioms listed above are canonical.

The logic has the *finite model property*:

FMP via filtration

Given a finite set of formulas Σ (closed under subformulas) and a model M, we define a preorder

$$x \preceq y \quad \text{iff} \quad (\forall \varphi \in \Sigma)\; x \Vdash \varphi \longrightarrow y \Vdash \varphi$$

and an equivalence relation

$$x \equiv y \quad \text{iff} \quad x \preceq y \wedge y \preceq x.$$

We define a new model on $\{[x] \mid x \in X\}$ with a valuation defined by:

$$[x] \Vdash p \quad \text{iff} \quad x \Vdash p.$$

FMP via filtration

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On $\equiv$-equivalence classes we define the partial order and monotone relations as follows:

$[x] \leq [y]$	$\Leftrightarrow$	$x \preceq y$
$[x] C [y]$	$\Leftrightarrow$	$x \preceq x' C y' \succeq y$
$[x] S^b [y] \wedge$ $[z] S^b [y]$	$\Leftrightarrow$	$[x] C [z] \ \wedge \ x \preceq x' S^b y' \preceq y$ $\wedge \ z \preceq z' S^b y'' \preceq y$
$[x] S^k [y]$	$\Leftrightarrow$	$x \preceq x' S^k y' \preceq y$ $(\wedge \ [x] C [y])$

Remark: all the properties of relations mentioned above are preserved, except of S^k-density, when the blue condition is present.

$$(\forall \varphi \in \Sigma) \ [x] \Vdash \varphi \ \text{ iff } \ x \Vdash \varphi.$$

Display calculus over (bi-)intuitionistic base

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$$\frac{X \vdash \sharp\varphi}{X \vdash \neg\varphi} \qquad \frac{X \vdash \varphi}{\neg\varphi \vdash \sharp X} \qquad \frac{X \vdash \sharp Y}{Y \vdash \sharp X}$$

$$\frac{X \vdash \varphi}{\bullet^b X \vdash \langle b \rangle \varphi} \qquad \frac{\bullet^b \varphi \vdash X}{\langle b \rangle \varphi \vdash X} \qquad \frac{\bullet^b X \vdash Y}{X \vdash \circ^b Y}$$

$$\frac{X \vdash \varphi}{\bullet^k X \vdash \langle k \rangle \varphi} \qquad \frac{\bullet^k \varphi \vdash X}{\langle k \rangle \varphi \vdash X} \qquad \frac{\bullet^k X \vdash Y}{X \vdash \circ^k Y}$$

Axioms via Structural rules

$$\frac{X \vdash Y}{\bullet^k X \vdash Y} \qquad \frac{\bullet^b X \vdash Y}{\bullet^k X \vdash Y} \qquad \frac{\bullet^b \bullet^k X \vdash Y}{\bullet^b X \vdash Y}$$

$\langle k \rangle \varphi \to \varphi$ $\qquad$ $\langle k \rangle \varphi \to \langle b \rangle \varphi$ $\qquad$ $\langle b \rangle \varphi \to \langle b \rangle \langle k \rangle \varphi$

$$\frac{X \vdash \sharp Y}{X \vdash \circ^b(\bullet^b Y > I)} \qquad \frac{X \vdash \sharp Y}{X \vdash \circ^k(\bullet^k Y > I)} \qquad \frac{X \vdash \sharp Y}{X \vdash \bullet^k Y > I}$$

$\langle b \rangle \varphi \wedge \langle b \rangle \neg \varphi \to \bot$ $\qquad$ $\langle k \rangle \varphi \wedge \langle k \rangle \neg \varphi \to \bot$ $\qquad$ $\langle k \rangle \varphi \wedge \neg \varphi \to \bot$

Modularity, completeness, cut elimination, subformula property.

Example of a proof

$$
\begin{array}{c}
p \vdash p \\
\hline
\neg p \vdash \sharp p \\
\hline
\neg p \vdash \circ^b(\bullet^b p > I) \\
\hline
\bullet^b \neg p \vdash \bullet^b p > I \\
\hline
\bullet^b p, \bullet^b \neg p \vdash I \\
\hline
\vdots \\
\hline
\langle b \rangle p, \langle b \rangle \neg p \vdash \bot \\
\hline
\langle b \rangle p \wedge \langle b \rangle \neg p \vdash \bot
\end{array}
$$

Common belief and knowledge

- The algebraic counterpart of the frame semantics (a complete lattice) $+$ Knaster-Tarski theorem $\Rightarrow$ fixed points.
- Common knowledge of φ can be expressed as the greatest fixed point

$$C\varphi \;\equiv\; \nu x. \bigwedge_{i\in I} \langle k\rangle_i(\varphi \wedge x).$$

- Obvious axiom and rule yield a non-compact logic, weak completeness remains an open problem.

Infinitary proof theory for ν

- One can turn to infinitary proof theory for fixed point logics, using finite approximations of fixed points.

$$\nu x^1.\alpha[x] = \alpha[\top] \quad \nu x^{n+1}.\alpha[x] = \alpha[\nu x^n.\alpha[x]]$$

- adopt axioms

$$\nu x.\alpha[x] \vdash \nu x^n.\alpha[x]$$

  and an infinitary rule

$$\{\nu x^n.\alpha[x] \mid n \in N\} \vdash \nu x.\alpha[x]$$

- consider the resulting Scott type consequence relation $\Gamma \vdash \Delta$ (Γ proves a finite disjunction of flas in Δ), and prove strong completeness via a canonical model

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THANK YOU!