On linear varieties of MTL-algebras

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MTL-algebras

An MTL-algebra is an algebra $\langle A, *, \rightarrow, \wedge, \vee, 0, 1\rangle$. such that:

1. $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded lattice with minimum 0 and maximum 1.
2. $\langle A, *, 1\rangle$ is a commutative monoid.
3. $\langle *, \rightarrow\rangle$ forms a *residuated pair*: $z * x \leq y$ iff $z \leq x \rightarrow y$ for all $x, y, z \in A$. In particular, it holds that $x \rightarrow y = \max\{z \in A : z * x \leq y\}$.
4. The following equation holds.

   (Prelinearity) $$(x \rightarrow y) \vee (y \rightarrow x) = 1.$$

   A totally ordered MTL-algebra is called *MTL-chain*.

- The class of MTL-algebras forms a variety, called $\mathbb{MTL}$. The logic corresponding to MTL-algebras is called MTL.
- An axiomatic extension of MTL is a logic obtained by adding other axioms to it.
- Every axiomatic extension of MTL is algebraizable in the sense of [Blok and Pigozzi, 1989], and hence every subvariety of $\mathbb{MTL}$ induces a logic.

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In this talk:

- We will study some general properties of linear varieties.
- We will classify all the linear varieties of BL-algebras.
- We will classify all the linear varieties of WNM-algebras.
- We will discuss a special case of linear varieties, the almost minimal varieties, providing a characterization theorem for the finite case.

Definition ([Montagna, 2011])

An axiomatic extension L of MTL has the *single chain completeness*, whenever there is an L-chain $\mathcal{A}$ such that L is complete w.r.t. it. In other terms, $\mathbb{L} = \mathbf{V}(\mathcal{A})$.

A first result

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Theorem

Let $\mathbb{L}$ be a variety of MTL-algebras. Then $\mathbb{L}$ is linear if and only if for every subvariety $\mathbb{L}'$ of $\mathbb{L}$ there is a chain $\mathcal{A} \in \mathbb{L}$ such that $\mathbb{L}' = \mathbb{V}(\mathcal{A})$, for some chain $\mathcal{A} \in \mathbb{L}$.

Theorem

Let $\mathbb{L}$ be a linear variety of MTL-algebras having the FMP, and containing at least an infinite chain. Then:

- *For every infinite chain $\mathcal{A} \in \mathbb{L}$, $\mathbf{V}(\mathcal{A}) = \mathbb{L}$.*
- *The only proper varieties of $\mathbb{L}$ are those generated by a finite chain.*
- *The order type of the lattice of the subvarieties of $\mathbb{L}$, ordered by inclusion, is $\omega + 1$.*
- *Let C be the class of all chains in $\mathbb{L}$. Then either every member of C is simple or every member of C is* bipartite*. This holds even if $\mathbb{L}$ contains only finite chains.*

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Interestingly, all the linear varieties that we found up to now have a lattice of subvarieties which is finite, or that has an order type of $\omega + 1$. This includes also the ones in $\mathbb{BL}$ and $\mathbb{WNM}$.

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There is a well know characterization theorem, for BL-chains.

Theorem ([Aglianò and Montagna, 2003])

Every BL-chain $\mathcal{A}$ can be uniquely decomposed as an ▸ ordinal sum $\bigoplus_{i \in I} \mathcal{W}_i$ *of totally ordered Wajsberg* ▸ hoops *whose first component $\mathcal{W}_{i_0}$ is bounded.*

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With $\mathbb{L}_k$ we will denote the variety generated by the k-element MV-chain $\mathbf{L}_k$, whose lattice reduct is $0 < \frac{1}{k-1} < \cdots \leq 1$.

Theorem

The linear subvarieties of $\mathbb{BL}$ *are exactly the following ones.*

- $\mathbb{G}$ *and* $\{\mathbb{G}_k\}_{k\geq 2}$.
- *The family of varieties* $\{\mathbb{L}_k : k-1 = h^n, 1 \leq h \text{ is prime and } n \geq 1\}$ *and* $\{\mathbf{V}(\mathbf{2} \oplus \mathbf{L}_k) : k-1 = h^n, 1 \leq h \text{ is prime and } n \geq 1\}$.
- *The variety* $\mathbb{C}$ *generated by Chang's MV-algebra.*
- $\mathbb{P}$ *(the variety of product algebras),* $\mathbb{P}_\infty$, *and* $\{\mathbb{P}_k\}_{k\geq 2}$.

Where:

- $\mathbb{P}_\infty$ *is the variety whose class of chains is given by all the chains of the form* $\mathbf{2} \oplus \bigoplus_{i\in I} \mathcal{C}_i$, *where every* $\mathcal{C}_i$ *is a cancellative hoop.*
- *For* $k \geq 2$, $\mathbb{P}_k$ *is the variety whose class of (non-trivial) chains is given by all the chains of the form* $\mathbf{2} \oplus \bigoplus_{i\in I} \mathcal{C}_i$, *where* $|I| \leq k$, *and every* $\mathcal{C}_i$ *is a cancellative hoop.*

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Let $\mathbb{L}$ be a variety of BL-algebras which is not n-contractive, for any n. Then $\mathbb{L}$ contains $\mathbb{P}$ or $\mathbb{C}$.

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By using the chain decomposition theorem it can be shown that the only linear varieties of BL-algebras being n-contractive are $\mathbb{G}$, $\{\mathbb{G}_k\}_{k\geq 2}$,
$\{\mathbb{L}_k : k-1 = h^n, 1 \leq h \text{ is prime and } n \geq 1\}$ and
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Assume now that $\mathbb{L}$ is linear, and $\mathbb{P} \subseteq \mathbb{L}$. Then the only possibility is that every (non-trivial) chain is $\mathbb{L}$ has the form $\mathbf{2} \oplus \bigoplus_{i\in I} \mathcal{C}_i$, where every $\mathcal{C}_i$ is a cancellative hoop.

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Then, a direct inspection shows that the only varieties with such properties are $\mathbb{P}$, $\mathbb{P}_\infty$, and $\{\mathbb{P}_k\}_{k\geq 2}$.

Linear varieties of BL-algebras - sketch of the proof

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Finally, if $\mathbb{L}$ is linear, and $\mathbb{C} \subseteq \mathbb{L}$, then necessarily $\mathbb{L} \subset \mathbb{MV}$. Using the Komori's classification, we can show that $\mathbb{L} = \mathbb{C}$.

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The operations $*, \rightarrow$ of a WNM-chain $\mathcal{A}$ are defined in the following way.

$$x * y = \begin{cases} 0 & \text{if } x \leq n(y), \\ \min\{x, y\} & \text{otherwise.} \end{cases} \qquad x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y, \\ \max\{n(x), y\} & \text{otherwise.} \end{cases}$$

Where $n: A \rightarrow A$ is a negation function, i.e. $n(1) = 0$, $n(n(x)) \geq x$, and if $x < y$, then $n(x) \geq n(y)$. A negation fixpoint is an element x such that $n(x) = x$.

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F-chains (with more than two elements) are those WNM-chains having a coatom c such that $n(c)$ is its predecessor, and $n(x) = c$, for every $0 < x < n(c)$.

Theorem

The linear subvarieties of $\mathbb{WNM}$ *are exactly the following ones.*

- $\mathbb{G}$ *and its subvarieties.*
- $\mathbb{DP}$ *and its subvarieties.*
- $\mathbb{NM}^-$ *and its subvarieties.*
- $\mathbb{F}$ *and its subvarieties.*

In particular, the only proper subvarieties of $\mathbb{L} \in \{\mathbb{G}, \mathbb{DP}, \mathbb{NM}^-, \mathbb{F}\}$ *are the ones of the form* $\mathbf{V}(\mathcal{A})$*, where* $\mathcal{A}$ *is a finite chain in* $\mathbb{L}$*. Moreover, the order type of the lattice of subvarieties of* $\mathbb{L}$ *is* $\omega + 1$*.*

The fact that $\mathbb{G}, \mathbb{DP}, \mathbb{NM}^-, \mathbb{F}$ are linear is a consequence of the way in which the operations $*$ and $\to$ are defined.

If a chain belongs to $\mathbb{WNM} \setminus \{\mathbb{G} \cup \mathbb{DP} \cup \mathbb{NM}^- \cup \mathbb{F}\}$, then we can show that it generates a non-linear variety, using the following theorem:

Theorem

- *Let $\mathcal{A}$ be a WNM-chain having an element $0 < x < 1$ with $\sim x = 0$. If $\mathcal{A} \notin \mathbb{G}$, then $\mathbf{V}(\mathcal{A})$ is not linear.*
- *Let $\mathcal{A}$ be a WNM-chain with a negation fixpoint. If $|A| > 3$ and $\mathcal{A}$ is not a DP-chain, then $\mathbf{V}(\mathcal{A})$ is not linear.*
- *Let $\mathcal{A}$ be a WNM-chain such that there is $0 < x < 1$ with $\sim\sim x = x$ and $\sim x \neq x$. If $\mathcal{A} \notin \mathbb{NM} \cup \mathbb{F}$, then $\mathbf{V}(\mathcal{A})$ is not linear.*

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Clearly, every almost minimal variety $\mathbb{L}$ of MTL-algebras is linear, and hence we have the following Corollary.

Corollary

- *The almost minimal varieties in* $\mathbb{BL}$ *are* $\mathbb{G}_3$, $\mathbb{P}$, $\mathbb{C}$, *and* $\{\mathbb{L}_k : k > 2 \text{ and } k-1 \text{ is prime}\}$.
- *The almost minimal varieties in* $\mathbb{WNM}$ *are* $\mathbb{G}_3$, $\mathbb{L}_3$, $\mathbb{NM}_4$.

Theorem

Let $\mathbb{L}$ be an almost minimal variety of MTL-algebras. Then, every chain in $\mathbb{L}$ such is simple or every chain in $\mathbb{L}$ is bipartite.

Almost minimal varieties - some theorems

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Let $\mathbb{L}$ be an almost minimal variety of MTL-algebras. Then, every chain in $\mathbb{L}$ such is simple or every chain in $\mathbb{L}$ is bipartite.

Theorem (Characterization of generic chains, finite case)

Given a finite MTL-chain $\mathcal{A}$, let $\mathbb{L} = \mathbf{V}(\mathcal{A})$. Then $\mathbb{L}$ is almost minimal if and only if $|\mathcal{A}| > 2$, and every element $a \in \mathcal{A} \setminus \{0,1\}$ generates $\mathcal{A}$.

Open problems

- Is it true that given a linear variety $\mathbb{L}$, every chain in $\mathbb{L}$ is linear or every chain in $\mathbb{L}$ is bipartite?

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- How about the almost minimal varieties of MTL-algebras? Is there a way to classify them, at least in the finite case?
- How about the computational complexity and the first-order case?

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APPENDIX

Definition

A *semihoop* is a structure $\mathcal{A} = \langle A, *, \sqcap, \Rightarrow, 1\rangle$ such that $\langle A, \sqcap, 1\rangle$ is an inf-semilattice with upper bound 1, $*$ is a binary operation on A with unit 1, and $\Rightarrow$ is a binary operation such that:

- $x \leq y$ iff $x \Rightarrow y = 1$,
- $(x * y) \Rightarrow z = x \Rightarrow (y \Rightarrow z)$.

A *bounded* semihoop is a semihoop with a minimum element; conversely, an *unbounded* hoop is a hoop without minimum.

- A hoop is a semihoop satisfying $x * (x \Rightarrow y) = y * (y \Rightarrow x)$.
- A Wajsberg hoop is a hoop satisfying $x \Rightarrow (x \Rightarrow y) = y \Rightarrow (y \Rightarrow x)$.

back

- Let $\langle I, \leq \rangle$ be a totally ordered set with minimum 0. For all $i \in I$, let $\mathcal{A}_i$ be a totally ordered semihoop such that for $i \neq j$, $A_i \cap A_j = \{1\}$, and assume that $\mathcal{A}_0$ is bounded.
- Then $\bigoplus_{i \in I} \mathcal{A}_i$ (the *ordinal sum* of the family $(\mathcal{A}_i)_{i \in I}$) is the structure whose base set is $\bigcup_{i \in I} A_i$, whose bottom is the minimum of $\mathcal{A}_0$, whose top is 1, and whose operations are

$$x \to y = \begin{cases} x \to^{\mathcal{A}_i} y & \text{if } x, y \in A_i \\ y & \text{if } \exists i > j (x \in A_i \text{ and } y \in A_j) \\ 1 & \text{if } \exists i < j (x \in A_i \setminus \{1\} \text{ and } y \in A_j) \end{cases}$$

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- As a consequence, if $x \in A_i \setminus \{1\}$, $y \in A_j$ and $i < j$ then $x < y$.

back

Definition

A variety $\mathbb{L}$ of MTL-algebras is said to be n-contractive ($n \geq 2$), whenever $L \models x^n = x^{n-1}$.

Theorem ([Bianchi and Montagna, 2011])

Every n-contractive BL-chain is isomorphic to an ordinal sum of finite MV-chains, each of them having at most n elements.

back

Axiomatization of MTL

The basic connective are $\{\wedge, \&, \rightarrow, \bot\}$ (formulas built inductively: a theory is a set of formulas). Useful derived connectives are the following ones:

(negation) $\neg\varphi \stackrel{\text{def}}{=} \varphi \rightarrow \bot$

(disjunction) $\varphi \vee \psi \stackrel{\text{def}}{=} ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$

(top) $\top \stackrel{\text{def}}{=} \neg\bot$

MTL can be axiomatized by using these axioms and modus ponens: $\frac{\varphi \quad \varphi\rightarrow\psi}{\psi}$.

(A1) $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$

(A2) $(\varphi \& \psi) \rightarrow \varphi$

(A3) $(\varphi \& \psi) \rightarrow (\psi \& \varphi)$

(A4) $(\varphi \wedge \psi) \rightarrow \varphi$

(A5) $(\varphi \wedge \psi) \rightarrow (\psi \wedge \varphi)$

(A6) $(\varphi \& (\varphi \rightarrow \psi)) \rightarrow (\psi \wedge \varphi)$

(A7a) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \& \psi) \rightarrow \chi)$

(A7b) $((\varphi \& \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$

(A8) $((\varphi \rightarrow \psi) \rightarrow \chi) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \chi) \rightarrow \chi)$

(A9) $\bot \rightarrow \varphi$

Hoops

Definition ([Ferreirim, 1992, Blok and Ferreirim, 2000])

A *hoop* is a structure $\mathcal{A} = \langle A, *, \to, 1\rangle$ such that $\langle A, *, 1\rangle$ is a commutative monoid, and $\to$ is a binary operation such that

$$x \to x = 1, \qquad x \to (y \to z) = (x * y) \to z \qquad \text{and} \qquad x * (x \to y) = y * (y \to x).$$

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- *A hoop is Wajsberg iff it satisfies the equation* $(x \to y) \to y = (y \to x) \to x$.

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- *A hoop is Wajsberg iff it satisfies the equation* $(x \to y) \to y = (y \to x) \to x$.
- *A hoop is cancellative iff it satisfies the equation* $x = y \to (x * y)$.
- *Totally ordered cancellative hoops coincide with unbounded totally ordered Wajsberg hoops, whereas bounded Wajsberg hoops coincide with (the 0-free reducts of) MV-algebras.*

back

Ordinal Sums

- Let $\langle I, \leq \rangle$ be a totally ordered set with minimum 0. For all $i \in I$, let $\mathcal{A}_i$ be a totally ordered Wajsberg hoop such that for $i \neq j$, $A_i \cap A_j = \{1\}$, and assume that $\mathcal{A}_0$ is bounded.

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A_j

$$x \to y = \begin{cases} x \to^{\mathcal{A}_i} y & \text{if } x, y \in A_i \\ y & \text{if } \exists i > j (x \in A_i \text{ and } y \in A_j) \\ 1 & \text{if } \exists i < j (x \in A_i \setminus \{1\} \text{ and } y \in A_j) \end{cases}$$

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- As a consequence, if $x \in A_i \setminus \{1\}$, $y \in A_j$ and $i < j$ then $x < y$.
- Note that, since every bounded Wajsberg hoop is the 0-free reduct of an MV-algebra, then the previous definition also works with these structures.

back

Bipartite MTL-algebras

Definition

- Given an MTL-chain $\mathcal{A}$, with $Rad(\mathcal{A})$ we denote the largest proper filter of $\mathcal{A}$.
- An MTL-chain $\mathcal{A}$ is said to be *bipartite* if $A = Rad(\mathcal{A}) \cup \overline{Rad}(\mathcal{A})$, where $\overline{Rad}(\mathcal{A}) = \{a \in A : \sim a \in Rad(\mathcal{A})\}$.

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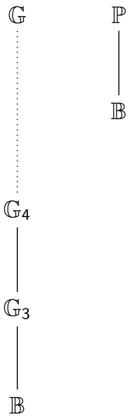
Theorem ([Noguera et al., 2005, Theorem 3.20])

Let $\mathcal{A}$ be an MTL-chain. Then the following conditions are equivalent:

- *$\mathcal{A}$ is bipartite.*
- *$Rad(\mathcal{A}) = A^+$ and $\mathcal{A}$ does not have a negation fixpoint.*
- *$\mathcal{A}/Rad(\mathcal{A}) \simeq \mathbf{2}$.*
- *$\mathcal{A}$ satisfies the following equation:*

$$(\mathrm{BP}_0) \qquad (\sim((\sim x)^2))^2 = \sim((\sim(x^2))^2).$$

back



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